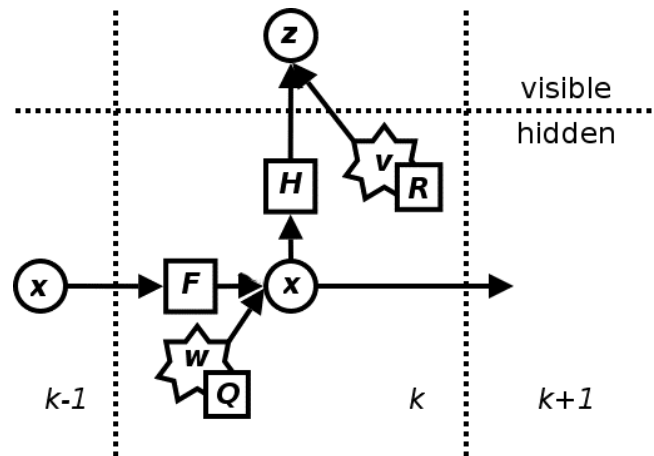


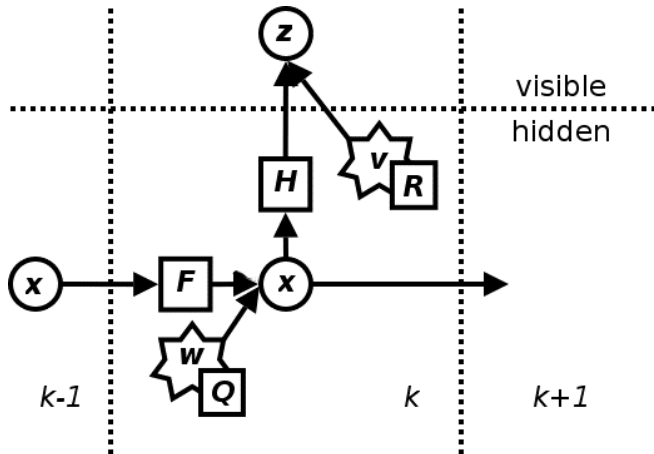
Kalman Filters



Derivation of Kalman Filter equations

Mark Fiala – CRV'09 Tutorial Day
May 24/2009

Kalman Filters



Predict

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}_k \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B}_k \mathbf{u}_k$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^T + \mathbf{Q}_k$$

Update

$$\mathbf{S}_k = \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^T + \mathbf{R}_k$$

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^T \mathbf{S}_k^{-1}$$

$$\tilde{\mathbf{y}}_k = \mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \tilde{\mathbf{y}}_k$$

$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1}$$

Normal Distribution - Gaussian



Make a donation to Wikipedia and give the gift of knowledge!

Log in / create account

article discussion edit this page history

Normal distribution

From Wikipedia, the free encyclopedia

In **probability theory** and **statistics**, the **normal distribution** or **Gaussian distribution** is a continuous **probability distribution** that describes data that clusters around a **mean** or average. The graph of the associated **probability density function** is bell-shaped, with a peak at the mean, and is known as the **Gaussian function** or **bell curve**.

The normal distribution can be used to describe, at least approximately, any **variable** that tends to cluster around the mean. For example, the heights of adult males in the United States are roughly normally distributed, with a mean of about 70 inches. Most men have a height close to the mean, though a small number of **outliers** have a height significantly above or below the mean. A **histogram** of male heights will appear similar to a bell curve, with the correspondence becoming closer if more data is used.

For theoretical reasons (such as the **central limit theorem**), any **variable** that is the sum of a large number of **independent factors** is likely to be normally distributed. For this reason, the normal distribution is used throughout **statistics**, **natural science**, and **social science**^[1] as a simple model for complex phenomena. For example, the **observational error** in an experiment is usually assumed to follow a normal distribution, and the **propagation of uncertainty** is computed using this assumption.

The probability density function for a normal distribution is given by the formula

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

where μ is the mean, σ is the **standard deviation** (a measure of the "width" of the bell), and exp denotes the **exponential function**. For a mean of 0 and a standard deviation of 1, this formula simplifies to

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2},$$

which is known as the **standard normal distribution**. When properly scaled and translated, the corresponding **cumulative distribution function** is known as the **error function**.

The Gaussian distribution is named for **Carl Friedrich Gauss**, who used it to analyze astronomical data^[2], and defined the formula for its probability density function.

Contents [hide]

1 History

navigation

- Main page
- Contents
- Featured content
- Current events
- Random article

search

Go Search

interaction

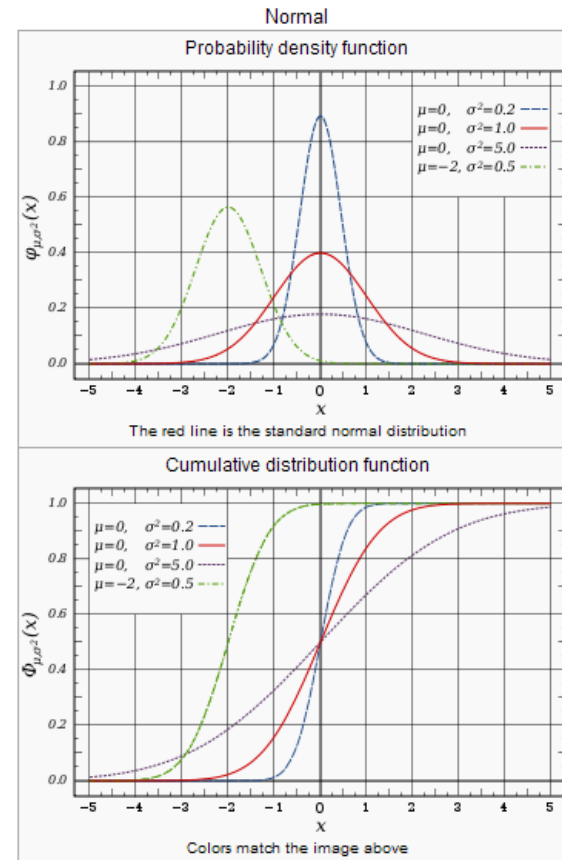
- About Wikipedia
- Community portal
- Recent changes
- Contact Wikipedia
- Donate to Wikipedia
- Help

toolbox

- What links here
- Related changes
- Upload file
- Special pages
- Printable version
- Permanent link
- Cite this page

languages

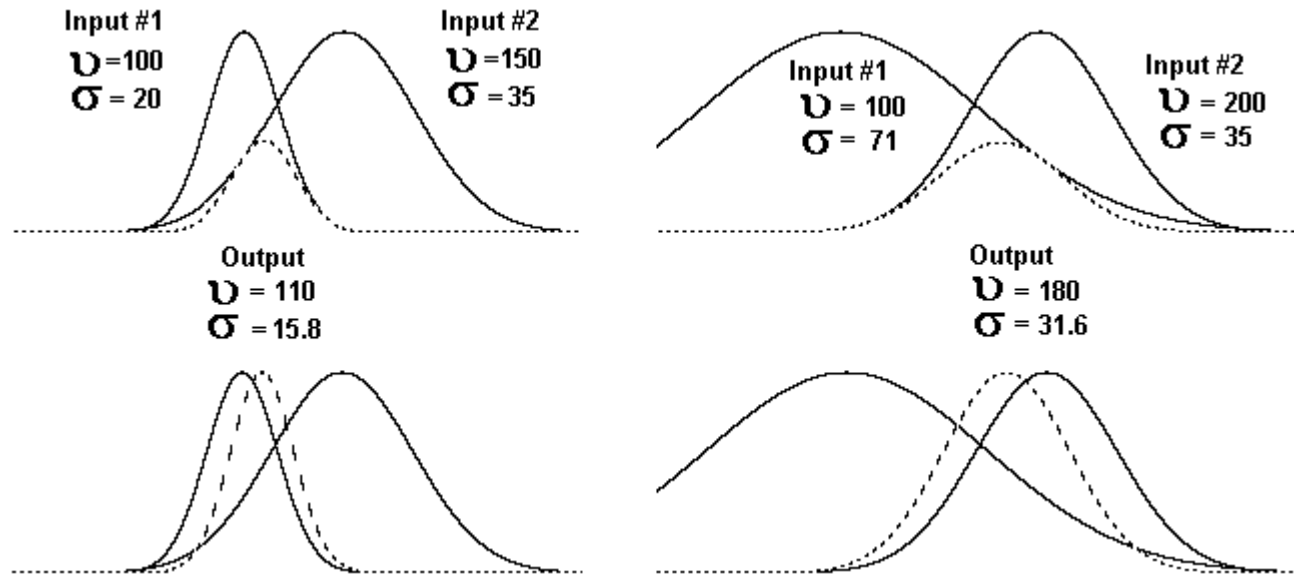
- العربية
- Azərbaycan



- Gaussian models normal distribution
- 1-D parameters: mean, sigma
- 2-D parameters: mean vector, covariance matrix

1-D Gaussians

- The Kalman filter is based on manipulating gaussian approximations of probability density functions (PDF's).
- Useful property of gaussian functions is that multiplying two gaussian functions yields a third gaussian.



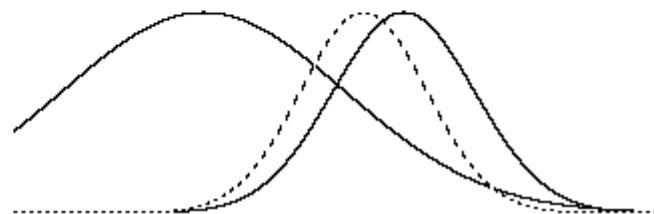
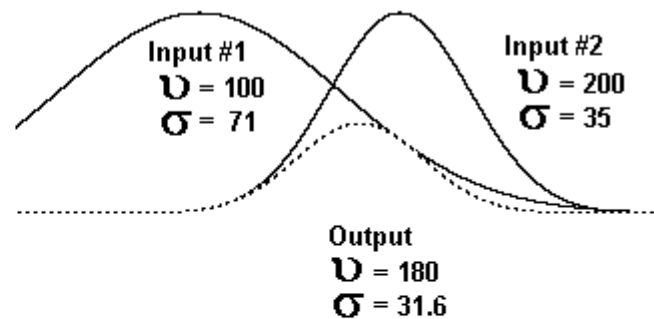
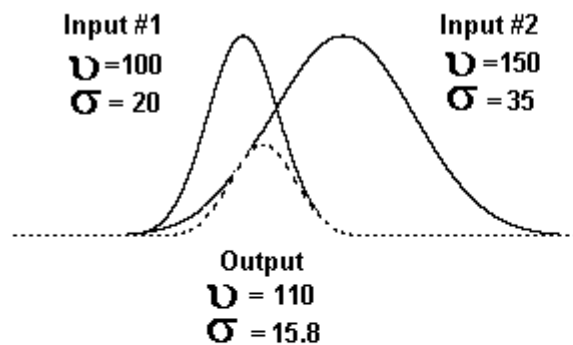
Hardware designers select one or the other based on situation, usually favor separate enables for timing reasons

Multiplying two 1-D Gaussians

$$PDF_1 = \exp\left(-\frac{(x-a)^2}{2e^2}\right), PDF_2 = \exp\left(-\frac{(x-b)^2}{2f^2}\right)$$

$$PDF_3 = PDF_1 \cdot PDF_2 = \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$\mu = a \frac{f^2}{e^2 + f^2} + b \frac{e^2}{e^2 + f^2}, \quad \sigma^2 = \frac{1}{\frac{1}{e^2} + \frac{1}{f^2}}$$



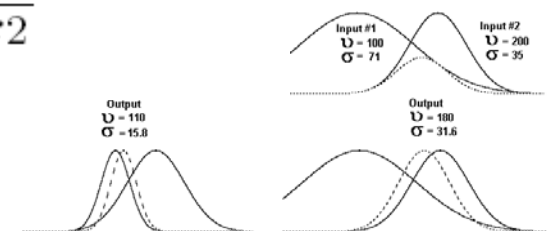
Deriving multiplying two 1-D Gaussians

$$\begin{aligned} \exp-\left(\frac{x-a}{2e^2}\right)^2 \exp-\left(\frac{x-b}{2f^2}\right)^2 &= \exp-\frac{1}{2}\left[\frac{(x-a)^2}{e^2} + \frac{(x-b)^2}{f^2}\right] \\ &= \exp-\frac{1}{2}\left[\frac{x^2 - 2ax - a^2}{e^2} + \frac{x^2 - 2bx + b^2}{f^2}\right] \\ &= \exp-\frac{1}{2}\left[\left(\frac{1}{e^2} + \frac{1}{f^2}\right)x^2 - 2\left(\frac{a}{e^2} + \frac{b}{f^2}\right)x + \left(\frac{a^2}{e^2} + \frac{b^2}{f^2}\right)\right] \\ &= \exp-\frac{1}{2}\left[\frac{x^2 - 2\mu x - \mu^2}{\sigma^2}\right] \end{aligned}$$

Matching terms, $\frac{1}{\sigma^2} = \left(\frac{1}{e^2} + \frac{1}{f^2}\right)$ and $\frac{\mu}{\sigma^2} = \left(\frac{a}{e^2} + \frac{b}{f^2}\right)$

$$\sigma^2 = \frac{1}{\frac{1}{e^2} + \frac{1}{f^2}}$$

$$\mu = \frac{1}{\frac{1}{e^2} + \frac{1}{f^2}} \left(\frac{a}{e^2} + \frac{b}{f^2}\right) = a \frac{f^2}{e^2 + f^2} + b \frac{e^2}{e^2 + f^2}$$



Deriving multiplying three 1-D Gaussians

$$\begin{aligned} & \exp - \left(\frac{x-a}{2e^2} \right)^2 \exp - \left(\frac{x-b}{2f^2} \right)^2 \exp - \left(\frac{x-c}{2g^2} \right)^2 \\ &= \exp - \frac{1}{2} \left[\left(\frac{x-a}{e^2} \right)^2 + \left(\frac{x-b}{f^2} \right)^2 + \left(\frac{x-c}{g^2} \right)^2 \right] \\ &= \exp - \frac{1}{2} \left[\frac{x^2 - 2ax - a^2}{e^2} + \frac{x^2 - 2bx + b^2}{f^2} + \frac{x^2 - 2cx + c^2}{g^2} \right] \\ &= \exp - \frac{1}{2} \left[\left(\frac{1}{e^2} + \frac{1}{f^2} + \frac{1}{g^2} \right) x^2 - 2 \left(\frac{a}{e^2} + \frac{b}{f^2} + \frac{c}{g^2} \right) x + \left(\frac{a^2}{e^2} + \frac{b^2}{f^2} + \frac{c^2}{g^2} \right) \right] \\ &= \exp - \frac{1}{2} \left[\frac{x^2 - 2\mu x - \mu^2}{\sigma^2} \right] \end{aligned}$$

Matching terms, $\frac{1}{\sigma^2} = \left(\frac{1}{e^2} + \frac{1}{f^2} \right)$ and $\frac{\mu}{\sigma^2} = \left(\frac{a}{e^2} + \frac{b}{f^2} \right)$

$$\sigma^2 = \frac{1}{\frac{1}{e^2} + \frac{1}{f^2}}$$

$$\mu = \frac{1}{\frac{1}{e^2} + \frac{1}{f^2}} \left(\frac{a}{e^2} + \frac{b}{f^2} \right) = a \frac{f^2}{e^2 + f^2} + b \frac{e^2}{e^2 + f^2}$$

Multiplying 1-D Gaussians

$$PDF_1 = \exp\left(-\frac{(x-a)^2}{2e^2}\right) \quad PDF_2 = \exp\left(-\frac{(x-b)^2}{2f^2}\right)$$
$$PDF_3 = \exp\left(-\frac{(x-c)^2}{2g^2}\right)$$

Multiplying 2 gaussians

$$PDF_1 \cdot PDF_2 = \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$\mu = a \frac{f^2}{e^2 + f^2} + b \frac{e^2}{e^2 + f^2}, \quad \sigma^2 = \frac{1}{\frac{1}{e^2} + \frac{1}{f^2}}$$

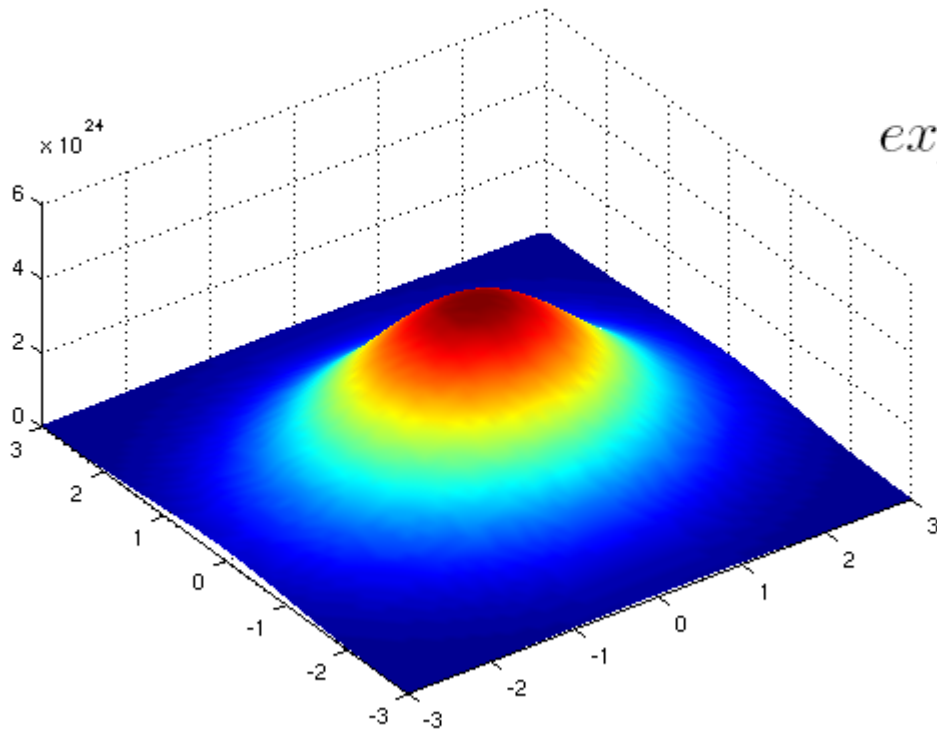
Multiplying 3 gaussians

$$\mu = a \frac{f^2 + g^2}{e^2 + f^2 + g^2} + b \frac{e^2 + g^2}{e^2 + f^2 + g^2} + c \frac{e^2 + f^2}{e^2 + f^2 + g^2},$$
$$\sigma^2 = \frac{1}{\frac{1}{e^2} + \frac{1}{f^2} + \frac{1}{g^2}}$$

Show 1d_gaussian_gui.exe

N-D Gaussians

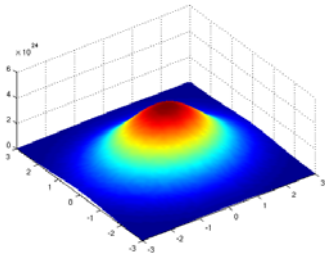
- Multi-dimensional gaussian can be created by putting gaussians on different orthogonal axes = multiplying with different variables
 - 2-D example: One gaussian in X-axis, one in Y-axis



$$\exp\left(-\frac{(x - \mu_x)^2}{2\sigma_x^2}\right) \cdot \exp\left(-\frac{(y - \mu_y)^2}{2\sigma_y^2}\right)$$

(Image from Wikipedia)

- 2-D gaussian aligned along X, Y axes



$$\begin{aligned}
 & \exp\left(-\frac{(x - \mu_x)^2}{2\sigma_x^2}\right) \cdot \exp\left(-\frac{(y - \mu_y)^2}{2\sigma_y^2}\right) \\
 &= \exp\left(-\frac{1}{2} \left(\frac{(x - \mu_x)^2}{\sigma_x^2} + \frac{(y - \mu_y)^2}{\sigma_y^2} \right)\right) \\
 &= \exp\left(-\frac{1}{2} \left(\begin{bmatrix} x - \mu_x & y - \mu_y \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_x^2} & 0 \\ 0 & \frac{1}{\sigma_y^2} \end{bmatrix} \begin{bmatrix} x - \mu_x \\ y - \mu_y \end{bmatrix} \right)\right) \\
 &= \exp\left(-\frac{1}{2} ([\mathbf{X} - \mathbf{U}]^t \Sigma^{-1} [\mathbf{X} - \mathbf{U}])\right)
 \end{aligned}$$

Where $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{U} = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$

$$\Sigma = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}, \Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_x^2} & 0 \\ 0 & \frac{1}{\sigma_y^2} \end{bmatrix}$$

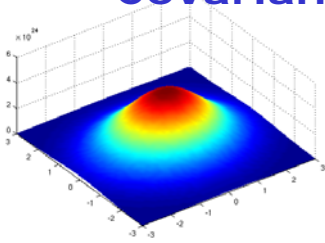
- Use rotation matrix \mathbf{R} to align along arbitrary axes

$$\begin{aligned}
 PDF &= \exp\left(-\frac{1}{2} ([\mathbf{X} - \mathbf{U}]^t \mathbf{R}^t \Sigma^{-1} \mathbf{R} [\mathbf{X} - \mathbf{U}])\right) \\
 &= \exp\left(-\frac{1}{2} ([\mathbf{X} - \mathbf{U}]^t \mathbf{C} [\mathbf{X} - \mathbf{U}])\right)
 \end{aligned}$$

- Matrix \mathbf{C} is not diagonal as is Σ

Justification of N-D gaussians

- Covariance matrix of an 2-D data set



$$Cov = \begin{bmatrix} \sum (x_i - \mu_x)(x_i - \mu_x) & \sum (x_i - \mu_x)(y_i - \mu_y) \\ \sum (x_i - \mu_x)(y_i - \mu_y) & \sum (y_i - \mu_y)(y_i - \mu_y) \end{bmatrix}$$

- SVD**

$A = UDV^t$ $D = \text{diagonal}$
 SVD of $A = S^t S$: $U = V$
 $A = UDU^t$

- Covariance matrix of an N-D data set

$$Cov = S^t S = R^t \Sigma R$$

$$Cov^{-1} = R^t \Sigma^{-1} R = C$$

$$\Sigma = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}, \Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_x^2} & 0 \\ 0 & \frac{1}{\sigma_y^2} \end{bmatrix}$$

- We need $1/\sigma^2$ form

Therefore

- a covariance matrix can be rotated to axes where Σ is a diagonal matrix
- a covariance matrix can be represented as an N-D shape of orthogonal gaussians

$$PDF = \exp - \frac{1}{2} ([\mathbf{X} - \mathbf{U}]^t \mathbf{R}^t \Sigma^{-1} \mathbf{R} [\mathbf{X} - \mathbf{U}])$$

$$= \exp - \frac{1}{2} ([\mathbf{X} - \mathbf{U}]^t \mathbf{C} [\mathbf{X} - \mathbf{U}])$$

- PDF=k =ellipse in 2-D, =ellipsoid in 3-D

Linear Functions and PDF's

What happens to a PDF when passing through a linear function (matrix operation)?

- Input variable X – has mean U_x and covariance Cov_x
- Linear function $Y = A X$
- Output variable Y – has mean U_y and covariance Cov_y

$$U_y = A U_x$$

$$Cov_y = A Cov_x A^t$$

$$\begin{aligned}
 PDF &= \exp -\frac{1}{2} ([\mathbf{X} - \mathbf{U}]^t \mathbf{R}^t \Sigma^{-1} \mathbf{R} [\mathbf{X} - \mathbf{U}]) \\
 &= \exp -\frac{1}{2} ([\mathbf{X} - \mathbf{U}]^t \mathbf{C} [\mathbf{X} - \mathbf{U}]) \\
 PDF &= \exp -\frac{1}{2} ([\mathbf{X}^t - \mathbf{U}^t] \mathbf{C} [\mathbf{X} - \mathbf{U}])
 \end{aligned}$$

- **Expanded form – useful for multiplication**

$$\begin{aligned}
 PDF &= \exp -\frac{1}{2} ([\mathbf{X}^t - \mathbf{U}^t] \mathbf{C} [\mathbf{X} - \mathbf{U}]) \\
 &= \exp -\frac{1}{2} ([\mathbf{X}^t \mathbf{C} - \mathbf{U}^t \mathbf{C}] [\mathbf{X} - \mathbf{U}]) \\
 &= \exp -\frac{1}{2} (\mathbf{X}^t \mathbf{C} \mathbf{X} - \mathbf{U}^t \mathbf{C} \mathbf{X} - \mathbf{X}^t \mathbf{C} \mathbf{U} + \mathbf{U}^t \mathbf{C} \mathbf{U}) \\
 &= \exp -\frac{1}{2} (\mathbf{X}^t \mathbf{C} \mathbf{X} - 2\mathbf{X}^t \mathbf{C} \mathbf{U} + \mathbf{U}^t \mathbf{C} \mathbf{U})
 \end{aligned}$$

Multiplying N-D Gaussians

Each gaussian PDF has a mean (centroid) vector \underline{U} and a covariance \underline{C}^{-1}

$$\text{Inputs} \quad PDF_1 = \exp\left[-\frac{1}{2}([\mathbf{X} - \mathbf{U}_1]^t \mathbf{C}_1 [\mathbf{X} - \mathbf{U}_1])\right] \quad PDF_2 = \exp\left[-\frac{1}{2}([\mathbf{X} - \mathbf{U}_2]^t \mathbf{C}_2 [\mathbf{X} - \mathbf{U}_2])\right]$$

$$\text{Output} \quad PDF_R = \exp\left[-\frac{1}{2}([\mathbf{X} - \mathbf{U}_r]^t \mathbf{C}_r [\mathbf{X} - \mathbf{U}_r])\right] = \exp\left[-\frac{1}{2}(\mathbf{X}^t \mathbf{C}_r \mathbf{X} - 2\mathbf{X}^t \mathbf{C}_r \mathbf{U}_r + \mathbf{U}_r^t \mathbf{C}_r \mathbf{U}_r)\right]$$

Multiplying 2 gaussians

$$\begin{aligned} PDF_1 \cdot PDF_2 &= \exp\left[-\frac{1}{2}(\mathbf{X}^t \mathbf{C}_1 \mathbf{X} - 2\mathbf{X}^t \mathbf{C}_1 \mathbf{U}_1 + \mathbf{U}_1^t \mathbf{C}_1 \mathbf{U}_1)\right] \cdot \exp\left[-\frac{1}{2}(\mathbf{X}^t \mathbf{C}_2 \mathbf{X} - 2\mathbf{X}^t \mathbf{C}_2 \mathbf{U}_2 + \mathbf{U}_2^t \mathbf{C}_2 \mathbf{U}_2)\right] \\ &= \exp\left[-\frac{1}{2}(\mathbf{X}^t \mathbf{C}_1 \mathbf{X} - 2\mathbf{X}^t \mathbf{C}_1 \mathbf{U}_1 + \mathbf{U}_1^t \mathbf{C}_1 \mathbf{U}_1 + \mathbf{X}^t \mathbf{C}_2 \mathbf{X} - 2\mathbf{X}^t \mathbf{C}_2 \mathbf{U}_2 + \mathbf{U}_2^t \mathbf{C}_2 \mathbf{U}_2)\right] \\ &= \exp\left[-\frac{1}{2}(\mathbf{X}^t [\mathbf{C}_1 + \mathbf{C}_2] \mathbf{X} - 2\mathbf{X}^t [\mathbf{C}_1 \mathbf{U}_1 + \mathbf{C}_2 \mathbf{U}_2] + \mathbf{U}_1^t \mathbf{C}_1 \mathbf{U}_1 + \mathbf{U}_2^t \mathbf{C}_2 \mathbf{U}_2)\right] \end{aligned}$$

$$\mathbf{C}_r = \mathbf{C}_1 + \mathbf{C}_2$$

$$\begin{aligned} \mathbf{C}_r \mathbf{U}_r &= \mathbf{C}_1 \mathbf{U}_1 + \mathbf{C}_2 \mathbf{U}_2, \\ \mathbf{U}_r &= \mathbf{C}_r^{-1} [\mathbf{C}_1 \mathbf{U}_1 + \mathbf{C}_2 \mathbf{U}_2] \end{aligned}$$

Multiplying N-D Gaussians

Each gaussian PDF has a mean (centroid) vector \underline{U} and a covariance \underline{C}^{-1}

$$\text{Inputs} \quad PDF_1 = \exp\left(-\frac{1}{2}([\mathbf{X} - \mathbf{U}_1]^t \mathbf{C}_1 [\mathbf{X} - \mathbf{U}_1])\right) \quad PDF_2 = \exp\left(-\frac{1}{2}([\mathbf{X} - \mathbf{U}_2]^t \mathbf{C}_2 [\mathbf{X} - \mathbf{U}_2])\right)$$

$$\text{Output} \quad PDF_R = \exp\left(-\frac{1}{2}([\mathbf{X} - \mathbf{U}_r]^t \mathbf{C}_r [\mathbf{X} - \mathbf{U}_r])\right) = \exp\left(-\frac{1}{2}(\mathbf{X}^t \mathbf{C}_r \mathbf{X} - 2\mathbf{X}^t \mathbf{C}_r \mathbf{U}_r + \mathbf{U}_r^t \mathbf{C}_r \mathbf{U}_r)\right)$$

$$\mathbf{C}_r = \mathbf{C}_1 + \mathbf{C}_2 \qquad \mathbf{U}_r = \mathbf{C}_r^{-1}[\mathbf{C}_1 \mathbf{U}_1 + \mathbf{C}_2 \mathbf{U}_2]$$

Rename covariance $\mathbf{P} = \mathbf{C}^{-1}$

$$\text{Inputs} \quad PDF_1 = \exp\left(-\frac{1}{2}([\mathbf{X} - \mathbf{U}_1]^t \mathbf{P}_1^{-1} [\mathbf{X} - \mathbf{U}_1])\right) \quad PDF_2 = \exp\left(-\frac{1}{2}([\mathbf{X} - \mathbf{U}_2]^t \mathbf{P}_2^{-1} [\mathbf{X} - \mathbf{U}_2])\right)$$

$$\text{Output} \quad PDF_R = PDF_1 \cdot PDF_2 = \exp\left(-\frac{1}{2}([\mathbf{X} - \mathbf{U}_r]^t \mathbf{P}_r^{-1} [\mathbf{X} - \mathbf{U}_r])\right)$$

$$\mathbf{P}_r = (\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1})^{-1} \qquad \mathbf{U}_r = \mathbf{P}_r(\mathbf{P}_1^{-1} \mathbf{U}_1 + \mathbf{P}_2^{-1} \mathbf{U}_2)$$

Multiplying Gaussians

Multiplying 1-D gaussians

$$PDF_1 = \exp\left(-\frac{(x-a)^2}{2e^2}\right) \quad PDF_2 = \exp\left(-\frac{(x-b)^2}{2f^2}\right)$$

$$PDF_3 = \exp\left(-\frac{(x-c)^2}{2g^2}\right)$$

$$PDF_1 \cdot PDF_2 = \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$\mu = a \frac{f^2}{e^2 + f^2} + b \frac{e^2}{e^2 + f^2}, \quad \sigma^2 = \frac{1}{\frac{1}{e^2} + \frac{1}{f^2}}$$

Multiplying N-D gaussians

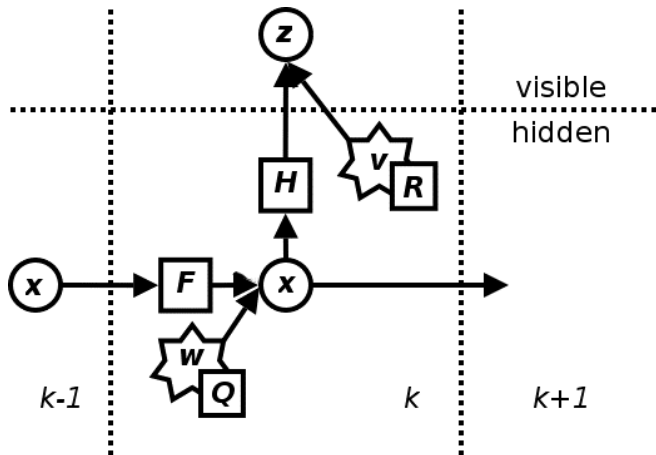
Inputs $PDF_1 = \exp\left(-\frac{1}{2}([\mathbf{X} - \mathbf{U}_1]^t \mathbf{P}_1^{-1} [\mathbf{X} - \mathbf{U}_1])\right) \quad PDF_2 = \exp\left(-\frac{1}{2}([\mathbf{X} - \mathbf{U}_2]^t \mathbf{P}_2^{-1} [\mathbf{X} - \mathbf{U}_2])\right)$

Output $PDF_R = PDF_1 \cdot PDF_2 = \exp\left(-\frac{1}{2}([\mathbf{X} - \mathbf{U}_r]^t \mathbf{P}_r^{-1} [\mathbf{X} - \mathbf{U}_r])\right)$

$$\mathbf{P}_r = (\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1})^{-1}$$

$$\mathbf{U}_r = \mathbf{P}_r(\mathbf{P}_1^{-1} \mathbf{U}_1 + \mathbf{P}_2^{-1} \mathbf{U}_2)$$

Kalman Filters



Predict

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}_k \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B}_k \mathbf{u}_k$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^T + \mathbf{Q}_k$$

Update

$$\mathbf{S}_k = \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^T + \mathbf{R}_k$$

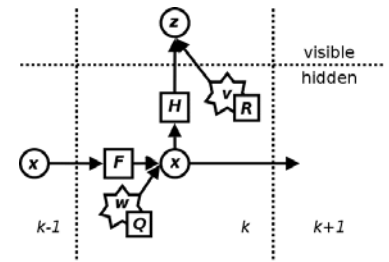
$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^T \mathbf{S}_k^{-1}$$

$$\tilde{\mathbf{y}}_k = \mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \tilde{\mathbf{y}}_k$$

$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1}$$

Kalman Filters



Predict

$$\hat{\mathbf{x}}_{k|k-1} = \mathbf{F}_k \hat{\mathbf{x}}_{k-1|k-1} + \mathbf{B}_k \mathbf{u}_k$$

$$\mathbf{P}_{k|k-1} = \mathbf{F}_k \mathbf{P}_{k-1|k-1} \mathbf{F}_k^T + \mathbf{Q}_k$$

Linear functions and PDF's

$$\begin{aligned} \mathbf{U}_y &= \mathbf{A} \mathbf{U}_x \\ \text{Cov}_y &= \mathbf{A} \text{Cov}_x \mathbf{A}^t \end{aligned}$$

Update

$$\mathbf{S}_k = \mathbf{H}_k \mathbf{P}_{k|k-1} \mathbf{H}_k^T + \mathbf{R}_k$$

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^T \mathbf{S}_k^{-1}$$

$$\tilde{\mathbf{y}}_k = \mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \tilde{\mathbf{y}}_k$$

$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1}$$

Linear functions and PDF's

$$\begin{aligned} \mathbf{U}_y &= \mathbf{A} \mathbf{U}_x \\ \text{Cov}_y &= \mathbf{A} \text{Cov}_x \mathbf{A}^t \end{aligned}$$

Baye's Rule and Kalman Filters

Baye's Rule

- $\text{Prob}(A|B) \sim = \text{Prob}(A)\text{Prob}(B|A)$
- Output variable Y – has mean U_y and covariance Cov_y

Kalman Filter – Find $\text{Prob}(X)$ given measurements Z

- X =state variables, Z =measurements
- Want $\text{Prob}(X|Z)$
- $\text{Prob}(X|Z) \sim = \text{Prob}(X)\text{Prob}(Z|X)$ *(update eqns)*
- X has normal distribution PDF given by mean = \hat{X} and Covar = P
- What is probability of observed Z given X ?
- $P(Z|X)$ has mean = Z and Covar = S
$$S_k = H_k P_{k|k-1} H_k^T + R_k$$

Multiply two PDF's = Kalman Filter

Kalman Filter = Multiply two N-D Gaussians

Multiplying N-D gaussians

Inputs $PDF_1 = \exp -\frac{1}{2} ([\mathbf{X} - \mathbf{U}_1]^t \mathbf{P}_1^{-1} [\mathbf{X} - \mathbf{U}_1])$ $PDF_2 = \exp -\frac{1}{2} ([\mathbf{X} - \mathbf{U}_2]^t \mathbf{P}_2^{-1} [\mathbf{X} - \mathbf{U}_2])$

Output $PDF_R = PDF_1 \cdot PDF_2 = \exp -\frac{1}{2} ([\mathbf{X} - \mathbf{U}_r]^t \mathbf{P}_r^{-1} [\mathbf{X} - \mathbf{U}_r])$

$$\mathbf{P}_r = (\mathbf{P}_1^{-1} + \mathbf{P}_2^{-1})^{-1} \quad \mathbf{U}_r = \mathbf{P}_r (\mathbf{P}_1^{-1} \mathbf{U}_1 + \mathbf{P}_2^{-1} \mathbf{U}_2)^{-1}$$

Multiply two PDF's = Kalman Filter

- $\text{Prob}(\mathbf{X}|\mathbf{Z}) \sim \text{Prob}(\mathbf{X})\text{Prob}(\mathbf{Z}|\mathbf{X})$ (*update eqns*)
- $PDF_1 = \mathbf{X}$ has normal distribution PDF given by mean = $\hat{\mathbf{X}}$ and Covar = \mathbf{P}
- $PDF_2 = \mathbf{Z}$ has normal distribution PDF given by mean = \mathbf{Z} and Covar = $\mathbf{HPH}^t + \mathbf{R}$
- $PDF_r = PDF_1 PDF_2 =$ next iteration \mathbf{X}, \mathbf{P}
- $\mathbf{P}_r = \mathbf{P}_{\text{next}} = [\mathbf{P}^{-1} + (\mathbf{HPH}^t + \mathbf{R})]^{-1}$
- $\mathbf{U}_r = \mathbf{X}_{\text{next}} = [\mathbf{P}^{-1} + (\mathbf{HPH}^t + \mathbf{R})]^{-1} [\mathbf{P}^{-1} \hat{\mathbf{X}} + (\mathbf{HPH}^t + \mathbf{R})^{-1} \mathbf{Z}]$

Kalman Filter = Multiply two N-D Gaussians

Multiply two PDF's = Kalman Filter

- $\mathbf{P}_r = \mathbf{P}_{\text{next}} = [\mathbf{P}^{-1} + (\mathbf{H}\mathbf{P}\mathbf{H}^t + \mathbf{R})]^{-1}$
- $\mathbf{U}_r = \mathbf{X}_{\text{next}} = [\mathbf{P}^{-1} + (\mathbf{H}\mathbf{P}\mathbf{H}^t + \mathbf{R})]^{-1} [\mathbf{P}^{-1}\hat{\mathbf{X}} + (\mathbf{H}\mathbf{P}\mathbf{H}^t + \mathbf{R})^{-1} \mathbf{z}]$

After some algebra and use of *inversion lemma*

$$\mathbf{K}_k = \mathbf{P}_{k|k-1} \mathbf{H}_k^T \mathbf{S}_k^{-1}$$

$$\tilde{\mathbf{y}}_k = \mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_{k|k-1}$$

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k \tilde{\mathbf{y}}_k$$

$$\mathbf{P}_{k|k} = (\mathbf{I} - \mathbf{K}_k \mathbf{H}_k) \mathbf{P}_{k|k-1}$$

EKF: Extended Kalman filter

- Allow non-linear functions (F, H)
- Apply functions to state $\mathbf{x}_k = f(\mathbf{x}_{k-1}, \mathbf{u}_k, \mathbf{w}_k)$
 $\mathbf{z}_k = h(\mathbf{x}_k, \mathbf{v}_k)$
- Apply jacobian to covariances
- Linearizing functions around current estimate