A COMPUTATIONAL THEORY OF 3D SHAPE RECONSTRUCTION FROM IMAGE CONTOURS

Ping Liang and John S. Todhunter

Dept. of Electrical Engineering
University of Pittsburgh, Pittsburgh, PA 15261, U.S.A.

ABSTRACT

The computational theory of 3D shape reconstruction from image contours proposed in this paper is based on the variational principles and has the theoretical framework suggested recently by Poggio [8]. By this theory, 3D shapes can be reconstructed from image contours by general physical constraint assumptions, namely the assumptions of minimum potential energy, isotropism and homogeneity of the material, and properly defined energy functionals. It is assumed that contours have been classified as surface discontinuity boundary contours, surface contours and extremal boundaries. Minimization of the energy functionals tends to maximize the symmetry and orthogonalize the surface junctions of the reconstructed object. Some early findings are obtained as the natural results of the theory. Theoretical developments and experimental results are successful and promising.

EXTENDED SUMMARY OF THE THEORY

One important function of early vision is the reconstruction of a 3D representation of a scene from 2D images. Stereopsis and structure from motion are the most explored in vision studies. Stereopsis and structure from motion require multiple images, but humans have the ability to perceive (with illusion) the 3D environment with only one eye or from a single picture.

There exist many sources of information about surfaces in an image such as texture, shading, shadow, etc. [9,12,13,17], but those methods are only applicable for certain special situations. It has been shown that shape reconstruction from contours is significantly more powerful than shape reconstruction from textures [7]. Barrow and Tenenbaum [11] argued that shape reconstruction from boundary contours is of fundamental importance in explaining surface perception and more important than shape reconstruction from shading. Steven [14] showed that surface contours also play an important role in shape reconstruction from image.

Theoretical studies [8] showed that the computational, ill-posed nature of early vision problems leads naturally to the application of the mathematical theory of regularizing ill-posed problems for solving them in terms of variational principles that enforce general physical constraints derived from a physical analysis of the problem. The constraints should be derived as a natural consequence of the physical laws governing the world we live in. The results of this research will not only impact our understanding of the early visual system in biological organisms, but also lead to development of computational algorithms and hardware designs for machine vision.

The fact that the human visual system has definite and consistent interpretations of contour images shows that it exploits some implicit assumption about the world. Without any knowledge of the nature of the process which generated the 3D shape, it is reasonable to assume that the given 2D contour is most likely to correspond to the projectively equivalent 3D shape with minimum potential energy. It is well known from classical mechanics that a physical system is stable if and only if its total potential energy is minimal. In many cases, it is also justified on the grounds that the surfaces tend to assume smooth and minimal energy configurations. Because there is no information available in the image contours about the material of the surface, the only reasonable assumption is that the surface material is isotropic and homogeneous. The computational theory of shape reconstruction from contours proposed in this paper is based on the variational principles in terms of general physical constraint assumptions : (1). the minimum energy principle; (2). isotropism and homogeneity of the material, i.e., the uniformity of the energy distribution.

To deform a system with minimum total potential energy to a non-minimum energy system, external energy has to be converted into potential energy in the system. As is well known, the differential equations describing a system in non-minimum energy state are far more complicated than the equations describing the system in its minimum energy state. Non-uniformity of energy distribution represents information about the system. Thus we can draw a correspondence between energy, energy distribution and information. Therefore the interpretation of the image contours by this theory is a minimum information interpretation in some sense.

Earlier studies such as [3,4,5,6,9,10,11,12,13,14,17] motivate and support the theory proposed in this paper. Barrow and Tenenbaum [11] optimized a smoothness measure to reconstruct planar curves and polyhedra. The optimization criterion used in [11] for continuous curves and straight lines are different, whereas a complete theory of shape reconstruction from contours should be able to accommodate both cases. The optimization criterion developed in [1] by the authors, which is a preliminary version of the theory proposed in this paper, uses a single underlying mechanism for both continuous curves and straight line contours. Witkin [12] developed a maximum likelihood approach to shape reconstruction from contours, and achieved some success in interpreting irregular shaped objects. This method is ineffective when the contour has a regular shape and does not compute the right slant of an ellipse. Brady and Yuille [7] developed an extremum principle which maximizes the ratio of the area to the square of the perimeter. Their method would be ineffective to curved surface and images with both boundary and surface contours. The theoretical framework proposed by Poggio [8] lends strong support to the theory proposed in this paper.

Kanade [9,17] developed a systematic method to recover 3D shapes from a single view by mapping image geometric properties into shape constraints. He proposed the assumption of mapping 2D skewed symmetry into 3D symmetry, and proved that the skewed symmetry can be a projection of real symmetry if and only if its surface gradient is on a certain hyperbola in the gradient space. We have proved that Kanade's assumption and hyperbola are natural results of the theory we proposed. Barnard [10] recently proposed a maximal orthogonal principle for 3D recovery based on psychophysical data. This principle is further developed and incorporated into our theory.

Some work has been done which offers proof to the minimum energy principle approach of shape reconstruction. Grimson [5,6] and Terzopoulos [3,4] used a thin plate model and constructed the 3D surfaces from the scattered stereo depth data by minimizing the total potential energy of the thin plate. The work by Barrow and Tenenbaum [11] is based on a similar idea in interpreting line drawings by optimizing a "smoothness" measure.

The outline of the theory is summarized as follows. It is assumed that the contours and junctions have been classified as surface discontinuity boundaries and junctions, surface contours, and extremal boundaries. The classification itself is a very important problem, and there is still no complete solution to it. Orthographical projection is assumed throughout.

First the reconstruction of a single surface from a simple closed 2D boundary contour is considered. Suppose that the 2D boundary contour has continuous curvature $\kappa_0(t)$ and is given by $r_0(t) = (\ x_0(t),\ y_0(t)\ ),\ t\varepsilon T = [a,b],\ x_0(a) = x_0(b),\ y_0(a) = y_0(b)$, where t is a parameter invariant under magnification or contraction. Backprojecting $r_0(t)$ into 3D as $r(t) = (\ x_0(t),\ y_0(t),\ z(t)\ )$ such that $r(t)$ has continuous curvature $\kappa(t)$ and torsion $\tau(t)$. Let

$$(\ \kappa(t)\frac{ds}{dt},\ \tau(t)\frac{ds}{dt}\ ) = (\ \kappa_t(t),\ \tau_t(t)\ ). \tag{1}$$

Define a vector

$$p(t)=\begin{cases}(\kappa_t(t),\ \tau_t(t),\kappa'_t(t),\tau'_t(t)), \text{ where } r_0(t) \text{ is convex.} \\ \\ (\pi-\kappa_t(t),\pi-\tau_t(t),\kappa'_t(t),\tau'_t(t)), \text{ otherwise.}\end{cases} \tag{2}$$

Suppose the derivatives $\kappa'_t,\ \tau'_t\varepsilon L^2$. Let $A = \{r(t)\,|\,r(t) = (\ x_0(t),\ y_0(t),\ z(t)\ ),\ \kappa(t),\ \tau(t) \text{ continuous},\ \kappa'_t,\ \tau'_t\varepsilon L^2\}$, $B = \{p(t)\,|\,r(t)\varepsilon A\}$. $C^0_c(\Omega)$ denotes the set of all continuous functions with compact support. Let H^m be the Sobolev spaces, and H^m_0 be the completion of $C^\infty_c(\Omega)$ in the norm $||\cdot||_{m,\Omega}$. Note that for $p(t)\varepsilon B,\ \kappa_t(t),\ \tau_t(t)\varepsilon H^1$.

Define an inner product as

$$(p_1, p_2) = \frac{k_\kappa}{2}\int_T \kappa_{t1}(t)\kappa_{t2}(t)dt + \frac{k_\tau}{2}\int_T \tau_{t1}(t)\tau_{t2}(t)dt + \frac{k_{\kappa 1}}{2}\int_T \kappa'_{t1}(t)\kappa'_{t2}(t)dt + \frac{k_{\tau 1}}{2}\int_T \tau'_{t1}(t)\tau'_{t2}(t)dt \tag{3}$$

where k_κ, k_τ are called energy factors of curvature and torsion, and $k_{\kappa 1}$, $k_{\tau 1}$ are called uniformity factors of curvature and torsion. The integrals are Lebesgue's integrals. When $p_1 = p_2$, the first two terms are measures of the potential energy in the reconstructed shape, and the last two terms are measures of the uniformity of the potential energy distribution.

The assumption that $\kappa_t(t),\ \tau_t\varepsilon H^1$ is a reasonable smoothness assumption of the curve can also be justified from the property of embedding $H^m(\Omega)$ into $C^j(\Omega)$. Suppose Ω is a subset of R^n, if $m>j+n/2$, then $H^m(\Omega)$ is embedded in $C^j(\Omega)$. For the surface case [3,4,5,6], the smoothness assumption $u\varepsilon H^2$ implies that $u\varepsilon C^0$. In the case of curves, $\kappa_t(t)$, $\tau_t(t)\varepsilon H^1$ implies that $\kappa_t(t),\ \tau_t(t)\varepsilon C^0$, i.e., the curve has continuous curvature and torsion.

Then the reconstruction of the surface is formulated as the following variational problems. The 2D contour $r_0(t)$ is first backprojected into 3D by minimizing

$$I_1(p(t)) = (\ p(t),\ p(t)\ ) \tag{4}$$

If the minimum is reached by $r^*(t)=(\ x_0(t),\ y_0(t),\ z^*(t)\ )$, then a surface u(x,y) is interpolated by minimizing

$$I_2(u) = \iint_\Omega \{\frac{1}{2}\ (\Delta u)^2 - (1-\sigma)(u_{xx}u_{yy} - u^2_{xy})\}\, dxdy \tag{5}$$

with the inhomogeneous Dirichlet boundary condition

$$u|_{\partial\Omega} = g(x,y) = z^* \tag{6}$$

Theorem 1: The energy measure $I_1(p)$ is invariant under linear transformation of the curve, i.e., if $r_2(t) = c\, r_1(t) + d$, then $I_1(p_1) = I_1(p_2)$.

This theorem is important in the reconstruction because all the similar shapes must have the same energy measure in order for a certain shape to reach the minimum regardless of its size. Consider the energy in an ellipse and a circle and suppose both of them are planar. If the energy measure is defined as

$$E(p(t)) = \frac{k_\kappa}{2} \int_L \kappa^2(s)\, ds \tag{7}$$

The energy in a circle is $\frac{k_\kappa}{2} \frac{2\pi}{R}$; the energy in an ellipse is $\frac{k_\kappa}{2} f(a, b)$, where a and b are the lengths of the two axes of the ellipse. So an ellipse with larger a and b will have less energy than a circle with a smaller R. This is the reason why by minimizing E, an ellipse cannot be interpreted as a circle [7]. By minimizing $I_1(p)$, an ellipse will be interpreted as a circle [1]. In implementation, $I_1(p)$ is easily discretized as

$$I_1(p(t)) = \frac{k_\kappa}{2} \sum_{i=1}^{N} Q_i^2 + \frac{k_\tau}{2} \sum_{i=1}^{N} P_i^2 \tag{8}$$

$$+ \frac{k_{\kappa 1}}{2} \sum_{i=1}^{N} (Q_{[i+1]} - Q_i)^2 + \frac{k_{\tau 1}}{2} \sum_{i=1}^{N} (P_{[i+1]} - P_i)^2$$

where Q_i is the external angle between the two successive sides of the approximating polygon, and P_i is the angle between the normals of the two planes determined by three successive sides of the approximating polygon. Where $[i+1] = (i+1)\mathrm{mod}(N)$.

Theorem 2: There exists an unique minimum value of $I_1(p)$, for all $p \varepsilon B$. Suppose the minimum is reached by curve $r^*(t)$, then $r^*(t)$ has continuous curvature and torsion.

The problem of minimizing $I_2(u)$ with an inhomogeneous Dirichlet boundary condition can be reformulated as follows. Suppose g is smooth enough, let $g \varepsilon H^2$, $v \varepsilon H_0^2$, then any $u = v + g \varepsilon H^2$ is in the admissible space. The problem becomes finding a $v \varepsilon H_0^2$ minimizing

$$I_2(u) = \frac{1}{2} a(u, u) - f(u) \tag{9}$$

$$= \frac{1}{2} a(v, v) - f(v) + a(v, g) + \frac{1}{2} a(g, g) - f(g)$$

with homogeneous boundary condition $v|_{\partial\Omega} = v_n|_{\partial\Omega} = 0$, which is equivalent to $u|_{\partial\Omega} = g$, $u_n|_{\partial\Omega} = g_n|_{\partial\Omega}$. Where a(u, v) is the energy inner product and $a(u, u) = I_2(u)$.

Theorem 3: There exists an unique solution $v \varepsilon U$, U is a subspace of H^2, which minimizes $I_2(u)$ with $u|_{\partial\Omega} = g$.

Next, the general cases are considered. Given boundary and surface contours with piecewise continuous $\kappa_0(t)$. Let the external jump angles of $\kappa(t)$ be α_i, i=1,2,...,n, the jump angles of $\tau(t)$ at surface discontinuity junctions be β_i, i=1,2,...,m. Then the reconstruction is formulated as backprojecting the contours into 3D by minimizing

$$J_1(p(t)) = \sum_i I_1(\, p_i(t)\,) + \frac{k_\beta}{2} \sum_i \left(\frac{\pi}{2} - \beta_i\right)^2 + \frac{k_\alpha}{2} \sum_i \alpha_i^2 \tag{10}$$

where k_β is the orthogonal link factor [1], k_a is the energy factor of curvature jump angles. The term of the jump angles in torsion (torsion jumps across surface discontinuity boundaries) $\sum_i \left(\frac{\pi}{2} - \beta_i\right)^2$, is part of the orthogonal links between surfaces based on the principle of maximal orthogonality between surfaces [1, 10]. Suppose $z^*(t)$ is the boundary and $C(x_i, y_i)$ are the points on the surface contours determined by minimizing $J_1(p(t))$. Then the surface is interpolated by minimizing

$$J_2(u) = \iint_\Omega \left\{\frac{1}{2} (\Delta u)^2 - (1-\sigma)(u_{xx} u_{yy} - u_{xy}^2)\right\} dxdy \tag{11}$$

$$+ \sum_i \left[\frac{\gamma_i}{2} (u(x_i, y_i) - c(x_i, y_i))\right]^2$$

$$u|_{\partial\Omega} = g(x, y) = z^*$$

The second term is interpreted as a set of vertical pins scattered inside Ω, the surface is only constrained by attaching ideal springs between those pin tips and the surface (see [3]). Where γ_i is the spring constant.

Again we have the similar results:

Theorem 1': The energy measure $J_1(p)$ is invariant under linear transformation of the curve.

Theorem 2': There exists an unique minimum value of $J_1(p)$, for all $p \varepsilon B$. Suppose the minimum is reached by $r^*(t)$, then $r^*(t)$ has piecewise continuous curvature and torsion.

Theorem 3': There exists an unique solution $v \varepsilon U$, U is a subspace of H^2, which minimizes $J_2(u)$ with $u|_{\partial\Omega} = g$.

Now we consider 3D shapes with more than one surface. Barnard [10] recently proposed a maximal orthogonality principle for 3D shape recovery based on psychophysical studies. This principle is modelled by putting ideal springs -- orthogonal links -- at the corners of the surface discontinuity boundaries. Note that in J_1 the term $\sum_i \left(\frac{\pi}{2} - \beta_i\right)^2$ is an orthogonal link term. So J_1 becomes

$$J_3 = J_1 + \sum_i \frac{k_\gamma}{2} (\frac{\pi}{2} - \gamma_i)^2 \quad (12)$$

where k_γ is the orthogonal link factor as in [1].

The finite element method is naturally suited to the problem of surface reconstruction from backprojected 3D contours because of the flexibility in the geometry of the method. Domains of complex shapes, boundary conditions, and nonuniform discretizations of the domain, all of which are features of the backprojected contours, can be easily handled in the finite element method.

Another possibility of dealing with the inhomogeneous Dirichlet boundary condition is by the penalty method. The boundary contours are treated same as surface contours, ideal springs are attached between the contours and the surface at a set of discrete points. Then this becomes a "free boundary" problem, and the solution is only unique upto a linear term, ax + by + c. To have an unique solution, there have to exist three noncollinear points on the contours to uniquely determine the linear term [3, 22]. This will always be satisfied in practice. When the spring constraints are strong enough, the solution would be close to the boundary.

For extremal boundaries, the normal to the boundary contours on the x-y plane is the normal to the surface. This can be handled by adding a penalty term to I_2, J_2,

$$\frac{k_n}{2} \sum_i (\frac{\nabla u(x_i, y_i)}{|\nabla u|} \Big|_{\partial\Omega} - \mathbf{n}(x_i, y_i))^2 \quad (13)$$

where $\mathbf{n}(x_i, y_i)$ are the unit normals to the extremal boundary contour on the x-y plane at points (x_i, y_i). The constraints are only at discrete points because of the consideration of implementation by the finite element method.

If the boundary consists of partly extremal boundary, partly surface discontinuity boundary, we will have a mixed boundary value problem. It can be treated accordingly.

Using the theory developed, we have proved that an ellipse will be interpreted as a circle, and skewed symmetric figures will be interpreted as real symmetry in 3D. Also several polyhedra and nonplanar polygon shapes have been successfully reconstructed.

Curved 3D shapes have also been successfully reconstructed from 2D contour images. The inputs to the programs are a set of 2D data points obtained by digitizing the contour drawings by a digitizer. $I_1(p)$ or $J_1(p)$ is minimized by the Levenberg-Marquardt algorithm to reconstruct the 3D shapes.

ACKNOWLEDGMENT

This research is supported in part by NIH Grant DE/NS01697-19.

REFERENCES

1. P. Liang, J. S. Todhunter, "Three Dimensional Shape Reconstruction from Minimum Energy Principle", *Proc. of 2nd Intl. Conf. on AI Application*, Miami, Florida, Dec. 1985. (Technical Memo 85-041, Apr. 1985, Pattern Recognition Lab. Univ. of Pittsburgh).

2. P. Liang, J. S. Todhunter, "A Computational Theory of 3D Shape Reconstruction from Image Contours", , Technical Memo 85-091, Sept. 1985, Pattern Recognition Lab., Univ. of Pittsburgh..

3. D. Terzopoulos, "Multilevel Computational Processes for Visual Surfaces Reconstruction", *Compt Vision, Graphics and Image Proc*, Vol. 24, 1983, pp. 52-96.

4. D. Terzopoulos, "Multilevel Reconstruction of Visual Surfaces: Variational Principles and Finite Element Method Representation", AI Memo 671, Artificial Intelligence Lab., MIT, 1983.

5. W.E. Grimson, "Surface Consistency Constraints in Vision", *Compt.Vision, Graphics, and Image Processing*, Vol. 24, 1983, pp. 28-51.

6. W.E. Grimson, *From Images to Surfaces: A Computational Study of the Human Early Visual System*, MIT Press, 1981.

7. M. Brady, A. Yuille, "An Extremum Principle for Shape from Contour", *IEEE Trans. on Pattern Analysis & Mach. Intell.*, Vol. PAMI-6, No. 3, May , 1984, pp. 288-301.

8. T. Poggio, "Early Vision: From Computational Structure to Algorithms and Parallel Hardware", *Compt.Vision, Graphics, and Image Processing*, Vol. 31, 1985, pp. 139-155.

9. T. Kanade, "Geometrical Aspects of Interpreting Image as a Three-Dimensional Scene", *Proc.of IEEE*, Vol. 71, No. 7, July , 1983, pp. 789-802.

10. S.T. Barnard, "Choosing a Basis for Perceptual Space", *Compt. Vision, Graphics and Image Processing*, Vol. 29, No. 1, 1985, pp. 87-99.

11. H.G. Barrow, J.M. Tenenbaum, "Interpreting Line Drawings as Three-Dimensional Surfaces", *Artificial Intell*, Vol. 17, 1981, pp. 75-116.

12. A. P. Witkin, "Recovering Surface Shape and Orientation from Texture", *Artificial Intell*, Vol. 17, 1981, pp. 17-47.

13. K. Ikeuchi, B. K. P. Horn, "Numerical Shape from Shading and Occluding Boundaries", *Artificial Intell*, Vol. 17, 1981, pp. 141-184.

14. K. A. Steven, "The Visual Interpretation of Surface Contours", *Artificial Intell*, Vol. 17, 1981, pp. 47-73.

15. D. Marr, T. Poggio, "A Theory of Human Stereo Vision", *Proc. Roy. Soc. London, B*, Vol. 207, 1980, pp. 187-217.

16. J. E. Mayhew, J. P. Frisby, "Psychological and Computational Studies towards a Theory of Human Stereopsis", *Artificial Intell*, Vol. 17, 1981, pp. 349–385.

17. T. Kanade, "Recovery of Three-Dimensional Shape of an Object from a Single View", *Artificial Intell*, Vol. 17, 1981, pp. 409–460.

18. W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York, 1974.

19. S. Agmon, *Lectures on Elliptic Boundary Value Problems*, Van Nostrand, Princeton, 1965.

20. P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, North-Holland, Amsterdam, 1978.

21. R. Courant, D. Hilbert, *Methods of Mathematical Physics*, Interscience, London, Vol. I, 1953.

22. G. Strang, G. J. Fix, *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs, 1973.

23. G. Fairweather, *Finite Element Galerkin Methods for Differential Equations*, Marcel Dekker, New York, 1978.

24. D. G. Luenberger, *Introduction to Linear and Nonlinear Programming*, Addison-Wesley, Reading, Mass, 1973.

25. L. Collatz, W. Wetterling, *Optimization Problems*, Springer-Verlag, New York, 1975.

26. D. Perkins, "Visual Discrimination between Rectangular and Nonrectangular Parallelopipeds", *Percept. Psychophys*, Vol. 12, No. 5, 1972, pp. 396–400.

27. W.H. Ittelson, *The Ames Demonstration in Perception*, Hafner, New York, 1968.

28. J.Thompson, G.Hunt, *A General Theory of Elastic Stability*, John Wiley & Sons, New York, 1973.

29. R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

30. S. L. Sobolev, *Applications of Functional Analysis in Mathematical Physics*, Vol. 7, Translation of Mathematical Monographs, American Mathematical Society, 1963.

31. H. Spath, *Spline Algorithms for Curves and Surfaces*, Utilitas Mathematica Publishing, Winnipeg, 1974.