GEOMETRIC UNWARPING FOR DIGITAL SUBTRACTION MAMMOGRAPHY

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ABSTRACT

Three dimensional (3D) mammograms taken some time apart could be subtracted to bring out changes that might be due to growing breast carcinomas. A geometric unwarping method for digital subtraction mammography is presented. This procedure is tested on simulated, warped 2D images. Experimental results and performance evaluations are given. We conclude that local registration methods are sufficiently more accurate than the global methods to warrant their increased programming complexity and computing time. Several aspects of image registration which need further investigation and improvement are also discussed.

KEYWORDS - Geometric unwarping, Triangulation, Triangular interpolant, Digital subtraction mammography.

I. Introduction

Breast cancer is approaching an epidemic rate of 1 out of every 10 women [1]. The means to prevent breast cancer has not yet been found. Detection and treatment of breast cancer at early stages is the only method with proven potential for lowering the death rate from this disease. If smaller tumors could be detected than is now possible by standard mammography, many more lives could be saved. New technologies, such as computed tomography (CT), magnetic resonance imaging (MRI), digital subtraction angiography (DSA), transillumination scanning, and ultrasound offer a wide selection for breast imaging [2]. None of these procedures are, however, expected to replace two dimensional, projection mammography as the first-line imaging technique for the detection and diagnosis of breast cancer, in their present forms [3]. Improving breast imaging modalities is understandably a great challenge to diagnostic radiology.

Breast tissue is soft, flexible, and changes shape over time. Contrast between soft tissues of the breast is inherently low and relatively minor changes in mammary structure can signify the presence of a malignant breast tumor. Two three-dimensional x-ray mammograms taken some time apart, e.g., at annual mammographic screenings, could be subtracted to bring out changes that might be due to a growing breast carcinoma [4]. Because of the differences in patient positioning, breast compression, dose parameter settings, change of equipment, etc., the difference image of two mammograms may contain considerable false contrast due to misregistration. Thus a geometric operation has to be performed to match several corresponding features in the 3D mammograms prior to subtraction. Therefore, it is necessary to first find the 3D transformation or "geometric warping" that relates an earlier 3D mammogram of a breast to a current 3D mammogram. (We will leave the production of 3D mammograms to future reports.)

If a sufficient number of corresponding features or "control points" in the 3D mammograms could be determined, then the mammograms could be accurately registered over their whole extent. Control points, i.e., features which are visible or can be defined unambiguously, and whose coordinates can be determined in both 3D mammograms, could then be used to drive the geometric unwarping algorithms. This paper will address optimal interpolation and extrapolation from the control points. The selection of control points and optimal pixel density interpolation algorithms will be covered in later papers.

The 2D image registration problem can be stated as follows: Given the coordinates for N pairs of control points in two images of the same scene, $\{(x_i, y_i), (u_i, v_i), i = 1, 2, \ldots, N\}$, determine the geometric unwarping functions

$$x = x(u,v)$$
$$y = y(u,v)$$

that will register the images. We will refer to the image with coordinates (x,y) as the reference image and the image with coordinates (u,v) as the warped image. Once the coordinates of all pixels in the warped image are determined, we apply an appropriate interpolation procedure to its density function. We thus obtain an image whose geometric coordinates are registered, an unwarped image. Then further digital manipulations, such as digital subtraction from the reference image, can be applied to these unwarped images.

Since the form and amount of geometric distortion between two images may not be known, investigators of image registration in the past have used polynomials to represent the transformation or mapping functions. Steiner and Kirby [5] have used polynomials of degree one. Nack [6] has used polynomials of degree two. Polynomials of degree three have been used to register Landsat images with maps [7]. Leckie [8] has used different polynomials of up to degree four to register airborne images and has determined the best degree polynomial that can register a pair of images. Adams *et al.* [9] presented hardware for geometric unwarping, in which a quadratic warping function is used to correct the pixel coordinates.

These investigations used a single transformation function to register the whole image. This approach may become too complicated to manage as the degree of the polynomial increases. Moreover, the resulting polynomial sometimes exhibits excessive spatial undulations. In many images, geometric distortions are mainly due to local factors such as atmospheric turbulence, sensor nonlinearity, breast compression (perhaps causing local slippage or nonuniform shear of loosely connected tissues), and so on. Therefore, geometric unwarping methods based on a collection of local transformation functions are potentially more accurate in these cases. Recently piecewise linear methods were proposed to replace simple global transformation functions: Goshtasby [10] used a number of local piecewise linear transformations, each tuned to map well in local neighborhoods; and then turned to piecewise cubic transformations [11]. However, these methods may not correct the distortion sufficiently in some cases, as in digital subtraction mammography. There is a clear need to develop efficacious algorithms to rectify these geometric distortions. For the sake of this paper, we will use two dimensional distortions. The ideas are expected to generalize to 3D.

In the field of Computer Aided Geometric Design, surface representation and approximation problems have been studied extensively [12]-[16]. There are three geometric building blocks for surfaces: rectangles, triangles and points. C^0, C^1 and C^2 rectangular and non-rectangular interpolants have been investigated and are currently used. In applications such as geometric unwarping, it is difficult or impracticable to restrict control points to those on rectangular grids. Interpolation approaches based on triangles or other methods have to be used. In these cases multi-stage methods have also been suggested [17],[18]. Part of the motivation for multi-stage interpolation is that some methods that apply directly to ungridded data give undesirable results or they are inefficient when the number of data points is large. On the other hand, many methods that are accurate and efficient only apply to data on regular grids.

II. Method and Procedure

Consider the determination of the geometric unwarping functions. Let D be a domain in the u-v plane, and suppose x and y are functions defined on D. Suppose we are given the coordinate values $x_i = x(u_i,v_i)$ of x and $y_i = y(u_i,v_i)$ of y at some set of control points (u_i,v_i) located in D, for $i = 1,2,\ldots,N$. Our problem is to find functions x^* and y^* defined on D which reasonably approximate x and y, respectively. The determination of the geometric transformation function x^* (and similarly y^*) consists of the following steps.

(1) Partition the convex hull of the set of control points into triangles by connecting the control points with line segments.

(2) Estimate partial derivatives of x^* with respect to u and v at each of the control points using the data values on either a set of nearby control points (a local method) or all of the control points (a global method).

(3) For an arbitrary point (u,v) in the convex hull of the set of the control points, determine which triangle contains the point (point-inclusion problem), and compute an interpolated value $x^*(u,v)$ using an appropriate interpolant over triangles. Capability of extrapolation for a point outside the convex hull should also be provided.

2.1 Triangulation

Given a set of N distinct points S, a triangulation which covers the convex hull of the data points can be constructed. In general, there exist many different triangulations of S. However, all possible triangulations have the same number of triangles and the same number of edges. The following observations were made by Lawson [19]:

$$n_t = 2N - n_b - 2 \le 2N$$
$$n_e = 3N - n_b - 3 \le 3N$$

where n_t is the number of triangles, n_b the number of boundary points, and n_e the number of edges.

In many applications it is desirable to have triangles as "equilateral" as possible. In most cases interpolation over equilateral triangles produces better results than that over thin (or obtuse) ones. The optimal triangulation can be obtained by initially creating an arbitrary triangulation of the data and then optimizing it by reassigning edges using various criteria. Lawson [19] has given three optimal criteria: the max-min angle criterion, circle criterion and Thiessen region criterion. In this paper, Lawson's triangulation algorithm with the max-min angle criterion was used.

Triangulation can also be done by recursive algorithms [20]-[22]. In computational geometry, Thiessen regions are also referred to as Delaunay, Dirichlet and Voronoi regions. The corresponding triangulation is also referred to as a Delaunay triangulation.

2.2 Estimation of partial derivatives

For C^0 interpolation methods only positional data about the control points are used to calculate the unwarping functions. There is a need to estimate the partial derivatives to approximate the bivariate functions if C^1 and C^2 or smoother methods are to be used. A simple method would be to fit a quadratic bivariate polynomial to point $\mathbf{P_0}$ and five of its nearest neighbors (not necessarily sharing an edge with $\mathbf{P_0}$) and then determine the partial derivatives of the resultant polynomial at point $\mathbf{P_0}$. By a weighted least squares technique more accurate estimations can be obtained for $n \geq 6$ points.

Akima [23] proposed the following approach. To estimate the partial derivatives at point $\mathbf{P_0}$ using $\mathbf{P_0}$ and its m nearest points $\mathbf{P_1}, \mathbf{P_2}, \ldots, \mathbf{P_m}$, form vector products $\mathbf{V_{ij}} = (\mathbf{P_0}-\mathbf{P_i}) \times (\mathbf{P_0}-\mathbf{P_j})$, with $\mathbf{P_i}$ and $\mathbf{P_j}$ being every possible combination of data points, where $\mathbf{P_1}, \mathbf{P_2}, \ldots, \mathbf{P_m}$ are arranged to be counterclockwise about point $\mathbf{P_0}$. The vector sum $\mathbf{V}$ of all $\mathbf{V_{ij}}$'s is calculated. Finally, the first derivatives are estimated from the slopes of a plane which is normal to the vector sum. In a later paper Akima [24] suggested the improvement by weighting the contribution of each triangle: a small weight was given to the contribution of a large triangle or a thin triangle when the vector sum was calculated. The improved Akima method for estimation of partial derivatives was used in this paper. For an overview of methods for estimation of partial derivatives, see Nielson and Franke [25]. Stead [26] made an experimental comparison of different estimation techniques.

2.3 Interpolation and extrapolation

Smooth, finite dimensional interpolants over triangles have been known for more than 20 years. Their practical uses have been as "finite elements." Many triangular interpolants are polynomial [23],[27],[28], while the Bernstein-Bezier representation [29] and triangular splines [30],[31] have also been reported.

The simplest C^0 interpolation over triangles can be stated as follows. Given three 3D points (x_i, y_i, z_i) for i=1,2 and 3, where z_i is the function value at point (x_i, y_i), determine parameters of the plane which pass through these points. For any point (x,y) which is inside the triangle, the value z can be calculated from this plane.

There are two widely known C^1 triangular interpolants: the 21 parameter quintic and the Clough-Tocher triangle. Akima [23] used the former interpolant for bivariate interpolation and surface fitting from irregularly distributed data. The proof of smoothness of the quintic interpolation along the side of the triangle was also given by Akima [23]. This interpolation method was tested here for the geometric unwarping problem. A detailed description of the Clough-Tocher interpolant is found in Goshtasby [11]. For a complete review on triangular interpolants, see Barnhill [32].

The capability of extrapolating outside the convex hull of data points is desirable. An exterior point lies in either a semi-infinite rectangle or a semi-infinite triangle defined by the lines which pass through the boundary data points and are perpendicular to the boundary edges (Fig.1). For the quintic approach, a polynomial which is quintic in the variable measured in the direction of the border line segment and quadratic in the distance from the line segment is used in the semi-infinite rectangle. A bivariate quadratic polynomial which smoothly connects to the two polynomials in the neighboring semi-infinite rectangles is used in the semi-infinite triangle. In the linear approximation approach, however, the plane which is just the extension of the border triangular plane, and the plane which passes the boundary data point and connects the neighboring rectangular planes, are used in the semi-infinite reactangle and semi-infinite triangle, respectively.

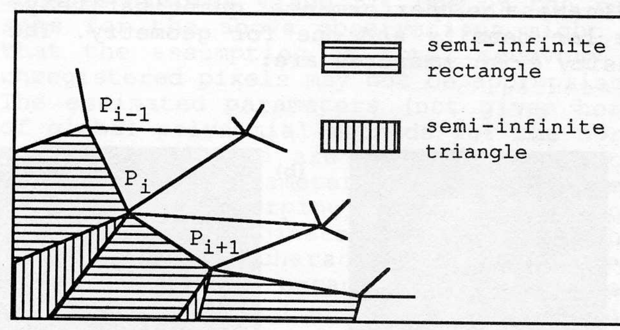

Fig.1. Partition of the exterior region into semi-infinite rectangles and triangles.

III. Results and Discussion

We use a Sony video camera with a Nikon microlens and a MacVision digitizer to digitize a portion of a mammogram. The mammogram is placed on an illuminator (view box). The video signal from the camera is digitized and stored on a Macintosh 512 kB computer. Our system has the capability of acquiring a 512 x 342 image with 8-bit resolution (256 gray levels). Image processing operations are performed by programs written in Pascal on a Macintosh II. Because of the current shortage of memory chips we restrict our operations on 64 x 64 images. Each pixel represents approximately 0.05 x 0.05 mm^2. The images are displayed on the Macintosh II and are photographed off the screen using a Nikon camera.

The reference image was taken from a digitized mammogram, and was transformed by using the following functions (see Goshtasby [10])

$$u = x + 3\sin(\pi x/32)$$

$$v = y - 3\sin(\pi y/32)$$

to get the warped image. Coordinates of control points in the reference image were chosen manually, approximately on an equilateral, triangular grid, and those in the warped image were calculated by the above functions. The reference and warped images are shown in Fig.2.

Unwarped images are obtained by the local quintic registration procedure. For comparison we show in Fig.3(a) the difference image of the reference and warped images. Figures 3(b) and 3(c) show difference images which are obtained by subtracting the unwarped images from the reference image, where linear and nearest neighbor resampling methods were used, respectively, during unwarping. The difference image of unwarping by local linear registration method and nearest neighbor rule is given in Fig.3(d). We also compare the proposed method with the global, polynomial interpolation methods. Parameters of linear, quadratic and cubic polynomials were obtained by using the least squares technique. The resultant difference images are shown in Figs.3(e)-(g).

Two types of error measures were used to evaluate the performance quantitatively: one for density and one for geometry. The density error measures are:

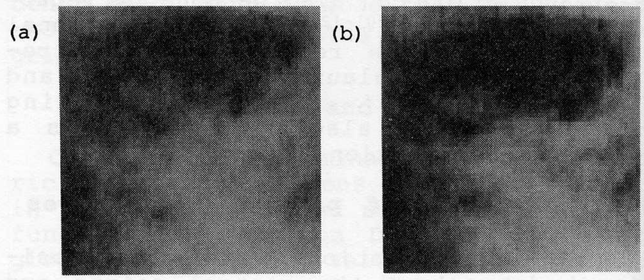

Fig.2. (a) The reference image. (b) The warped image obtained by using nearest neighbor density interpolation and the known warping functions. Pixel values of 159 and less are represented by black, 222 and greater by white, and others by gray shades.

$$\varepsilon_1 = \sqrt{\frac{1}{N}\sum_{i=1}^{N} |f_i - f_i^*|^2}$$

$$\varepsilon_2 = \frac{1}{N}\sum_{i=1}^{N} |f_i - f_i^*|$$

$$\varepsilon_3 = \max_{1 \le i \le N} |f_i - f_i^*|$$

where N is the total number of pixels in either image, and f_i and f_i^* represent the pixel density in the reference and unwarped images, respectively. Since the warping functions are known, we are able to calculate the geometric error measures. The following geometric error measures are ex-

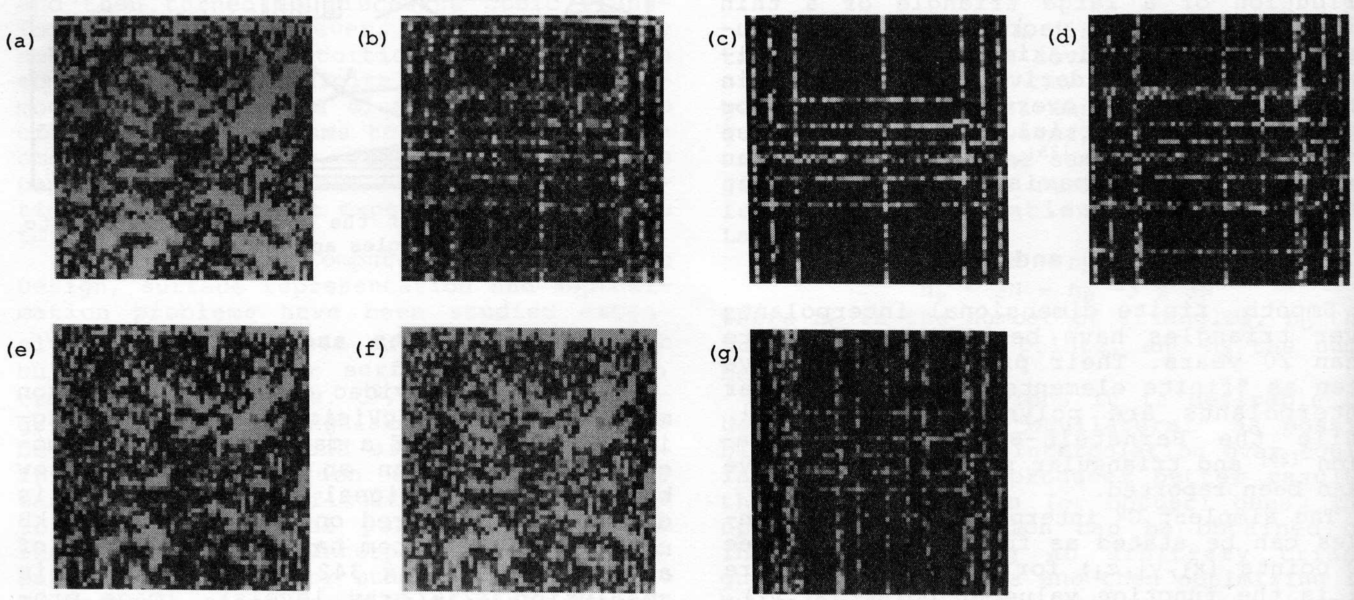

Fig.3. (a) The difference image of the reference and warped images. (b) The difference image for unwarping with bilinear density interpolation and local quintic registration. Difference images (c)-(g) obtained by using nearest neighbor interpolation and local quintic, local linear, least squares linear, least squares quadratic, and least squares cubic registrations, respectively. Difference of 0 is represented by black, 7 and greater by white, and others by gray shades.

pected to give a reasonable description of the performance of unwarping methods.

$$\delta_1 = \sqrt{\frac{1}{N}\sum_{i=1}^{N} d_i^{\,2}}$$

$$\delta_2 = \frac{1}{N}\sum_{i=1}^{N} d_i$$

$$\delta_3 = \max_{1 \le i \le N} d_i$$

where $d_i = ((u_i - u_i^*)^2 + (v_i - v_i^*)^2)^{1/2}$, N is the same as above, and (u_i, v_i) and (u_i^*, v_i^*) represent coordinates of pixels in the warped image which are determined by the known warping functions and numerical approaches, respectively. Note that these measures emphasize different aspects of image quality. A large difference in a few places causes the values of ε_1 and δ_1 to be large. The measures ε_2 and δ_2 emphasize the importance of many small errors rather than of a few large errors. The ε_3 and δ_3 are the largest difference between the reference and the unwarped images. These measures are shown in Tables 1-5.

From these measures we conclude that, as expected, the local quintic method is the best among these approaches. It can be seen from the geometric measures that the cubic least squares method produces better registration than the local linear technique. It is natural for this to happen because the specific warping functions used here do not have much local geometric distortion.

Table 1. Density measures with nearest neighbor interpolation

Methods	ε_1	ε_2	ε_3
Local quintic	1.439367	0.520020	12.0
Local linear	1.801421	0.804688	12.0
LS linear*	5.297129	3.828125	32.0
LS quadratic*	5.275607	3.809570	32.0
LS cubic	2.209819	1.085938	18.0

Table 2. Density measures with linear interpolation

Methods	ε_1	ε_2	ε_3
Local quintic	1.316957	0.778809	10.0
Local linear	1.542924	0.990967	11.0
LS linear*	4.838404	3.482910	32.0
LS quadratic*	4.809201	3.459473	32.0
LS cubic	1.853391	1.192383	15.0

* Note: While evaluating in these cases pixels which are not registered are discarded, in other words they are assumed exactly recovered.

Table 3. Geometric measures with nearest neighbor interpolation

Methods	δ_1	δ_2	δ_3
Local quintic	0.303706	0.284010	0.528869
Local linear	0.378976	0.344773	0.666999
LS linear	2.047397	1.951620	3.011342
LS quadratic	2.045149	1.949525	3.031876
LS cubic	0.322662	0.302871	0.639068

Table 4. Geometric measures with nearest neighbor interpolation

Methods	$\delta_1{}^*$	$\delta_2{}^*$	$\delta_3{}^*$
Local quintic	0.194894	0.158868	0.378672
Local linear	0.266010	0.201187	0.471640
LS linear	1.459559	1.300874	2.151875
LS quadratic	1.457852	1.299689	2.160514
LS cubic	0.218894	0.190155	0.443867

* Note: These measures are evaluated with the assumption that there is no vertical geometric distortion.

Table 5. Geometric measures with nearest neighbor interpolation

Methods	$\delta_1{}^{**}$	$\delta_2{}^{**}$	$\delta_3{}^{**}$
Local quintic	0.232924	0.207498	0.388004
Local linear	0.269930	0.231438	0.471639
LS linear	1.435796	1.281491	2.106565
LS quadratic	1.434331	1.280659	2.127076
LS cubic	0.237058	0.207819	0.459776

** Note: These measures are evaluated with the assumption that there is no horizontal geometric distortion.

From density measures (Tables 1 and 2) the local registration and cubic least squares with bilinear density interpolation give better results than those with the nearest neighbor rule. This conclusion is reversed for linear and quadratic least squares. It also seems that for local and cubic least squares registration methods the geometric unwarping in the horizontal coordinate is better than that in the vertical coordinate. But this is again reversed for linear and quadratic least squares (Tables 4 and 5). One of the reasons for the above observations might be that the assumption of total recovery of unregistered pixels may not be appropriate. The estimated parameters (not given here) of global polynomial methods for the horizontal coordinate are much closer to the corresponding parameters in the series expansion of the warping function than those for vertical coordinate. It seems that this is an inherent characteristic for these warping functions because a similar observation is obtained while using local approaches (see Tables 4 and 5).

IV. Conclusions

Local registration methods are more accurate than the global methods in cases where there may be much local geometric distortion to be unwarped, such as in digital subtraction mammography.

At this time much effort in computational geometry has been given to the two-dimensional point-inclusion (or point-location) problems, and they are well understood, while very little is known for 3D and even less for higher number of dimensions [22]. More efficient point-inclusion algorithms than we used can be incorporated to speed up the point locating procedure, which is a pre-processing step for surface fitting, and hence the interpolation procedures.

Research on three-dimensional interpolation from arbitrarily located data has been reported recently [33]-[35].

Algorithms for computing the Delaunay triangulation in n dimensional space ($n \geq 3$) have also been proposed [36],[37]. These advances are expected to facilitate further investigations on 3D image registration.

The prospective applications of this proposed image registration method go beyond mammography, and include image reconstruction from serial sections, remote and satellite image analysis, and image sequence analysis.

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