On Circles Recognition

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Abstract

This paper presents a criterion for deciding whether a set of lattice points is or not the digital image of a real circle or arc of a circle. For this purpose are used the digital-inscribable quadrilaterals, such that it is not necessarily a priori information about the center and radius of the circle. The set of lattice points is divided into digital-straight line segments. Then is verified if all three consecutive digital-straight line segments form a digital-inscribable quadrilateral (the fourth edge of the quadrilateral links the end points of this contour). This method permits a simple way of dividing the given set of lattice points into digital-straight line segments and arcs of digital circles.

Keywords: image processing, digitizing schemes, circles recognition, digital inscribable-quadrilaterals, contour decomposition.

1. Introduction

Image processing and pattern recognition for technical drawings are mainly concerned with straight line segments, texts and circles. We are interested in recognition of circles and arcs of circles. For this purpose are studied the following three problems:

1. A criterion to distinguish if a set of lattice points is the digital image of a real circle (recognition problem).

2. An algorithm for obtaining the center and radius of a real circle (or arc of a circle) whose digital image is equivalent with the given set of lattice points (approximation problem).

3. To divide the digital contour in segments such that each segment may be either a digital-straight line segment or a digital arc of a circle (segmentation problem).

In [6] Nakamura and Aizawa demonstrated a necessary and sufficient condition for a given set of lattice points to be the digital image of a real circle. This criterion, presented in section 3, needs the coordinates of the center of the circle.

The method presented in this paper tries to establish, first, if there exists a circle, preimage of a set of given lattice points, and only if the condition holds, to compute the center and radius of a preimage.

The organization of this paper includes five sections. The following section presents two schemes for digitizing planar curves, the third section, briefly, describes Nakamura's and Aizawa's results. Section 4 presents a necessary and sufficient condition for a set of lattice points to be a digital circle, using digital-inscribable quadrilaterals. The last section presents implementation results and conclusions.

2. Schemes for Digitizing Planar Curves

Consider a planar coordinate grid, such that all its nodes are lattice points. Let q be a curve in the plane. For the purpose of computer analysis it is usual to represent q by a finite set of lattice points, noted $I(q)$ and called the digital image of q. $I(q)$ can be obtained in various ways. In this section are defined the *grid digitization scheme* and *Kim's grid digitization scheme* for oriented planar arcs.

Definition 2.1 Grid digitization [2]

A set Q of lattice points is a digital image of an arc q, noted $Q = I(q)$, if the following conditions hold:

(i) If a lattice point $p = (h, k)$ is in Q, then q crosses a grid line at a point $z = (x, y)$ such that either $\max\{|h - x|, |k - y|\} < 1/2$ or $\max\{|h - x|, |k - y|\} = 1/2$ and p lies to the right of q.

(ii) If a lattice point $p = (h, k)$ is not in Q, then there is no crossing point $z = (x, y)$ that satisfies condition (i).

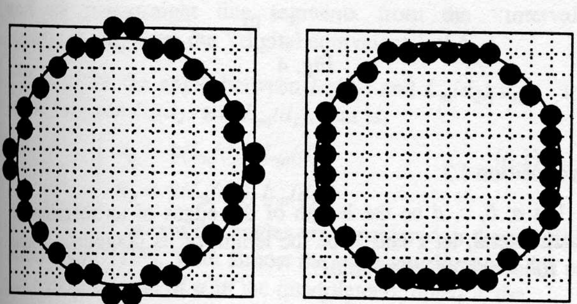
Definition 2.2 Kim's grid digitization [4]

A set Q of lattice points is a digital image of an arc q, noted $Q = IK(q)$, if the following conditions hold:

(i) If a lattice point $p = (h, k)$ is in Q, then q crosses a grid line at a point $z = (x, y)$ such that either $\max\{|h - x|, |k - y|\} < 1$ or $\max\{|h - x|, |k - y|\} = 1$ and p lies to the right of q.

(ii) If a lattice point $p = (h, k)$ is not in Q, then there is no crossing point $z = (x, y)$ that satisfies condition (i).

Fig. 1 shows the difference between the two digitization schemes . When using Kim's digitization scheme all points of $IK(q)$ lie in the interior of the real circle.



a) grid digitization b) Kim's grid digitization

Fig. 1

3. Conditions for Digital Images of Circles

Definition 3.1

A set of lattice points Q is called *a digital circle* if there is a real circle q such that $I(q) = Q$.

Definition 3.2

A set of lattice points A is called *an arc of a digital circle* if there is an arc a of a real circle, such that $I(a) = A$.

This paper refers both to digital circles and arcs of digital circles, but for a simpler expression, we shall note shortly, only "digital circles".

In this section is briefly presented a very important result of Nakamura and Aizawa [6], a necessary and sufficient condition for a set of lattice points to be a digital circle.

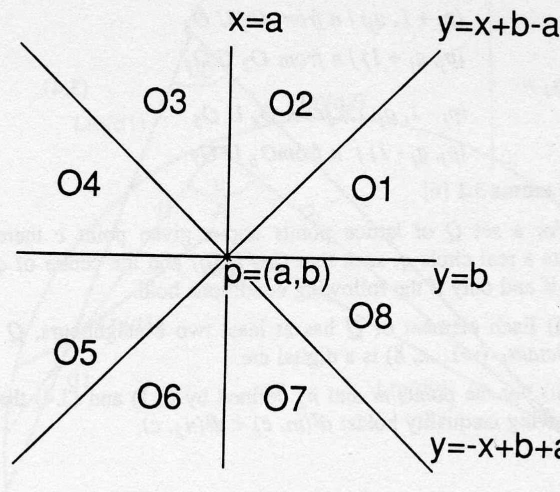


Fig. 2

Let $p = (a, b)$ be a point in the real plane. As shown in Fig. 2, the octants of the plane defined by p are the 8 angles bounded by the straight lines: $x = a$; $y = b$, $y = x + b - a$, $y = -x + b + a$.

Let Q be a finite set of lattice points, and let c be a given point in the real plane. Let $m = (p_i, q_i)$ and $n = (p_j, q_j)$ the lattice points defined as follows:

$$\begin{cases} d^2(m, c) = \max\{d^2(a, c) \mid a \text{ from } Q\} \\ d^2(n, c) = \min\{d^2(a, c) \mid a \text{ from } Q\} \end{cases} \quad (3.1)$$

where $d(.,.)$ is the Euclidian distance between the two points.

For these two points Nakamura and Aizawa defined two other points m_1 and n_1 as follows (see Fig. 3):

$$m_1 = \begin{cases} (p_i - 1/2, q_i) \mid m \text{ from } O_1 \cup O_8 \\ (p_i, q_i - 1/2) \mid m \text{ from } O_2 \cup O_3 \\ (p_i + 1/2, q_i) \mid m \text{ from } O_4 \cup O_5 \\ (p_i, q_i + 1/2) \mid m \text{ from } O_6 \cup O_7 \end{cases} \quad (3.2)$$

$$n_1 = \begin{cases} (p_j + 1/2, q_j) \mid n \text{ from } O_1 \cup O_8 \\ (p_j, q_j + 1/2) \mid n \text{ from } O_2 \cup O_3 \\ (p_j - 1/2, q_j) \mid n \text{ from } O_4 \cup O_5 \\ (p_j, q_j - 1/2) \mid n \text{ from } O_6 \cup O_7 \end{cases} \quad (3.3)$$

Lemma 3.1 [6]

For a set Q of lattice points and a given point c there exists a real circle q, such that $Q = I(q)$ and the center of q is c if and only if the following conditions hold:

(i) Each element of Q has at least two 8-neighbours, $Q \cap Octant_i$ (i = 1, ..., 8) is a digital arc.

(ii) For the points m_1 and n_1 defined by (3.1) (3.2) and (3.3) the following inequality holds: $d^2(m_1, c) < d^2(n_1, c)$.

Kim's digitization scheme defines point n_2 as follows:

$$n_2 = \begin{cases} (p_j + 1,\ q_j) \mid n \text{ from } O_1 \cup O_8 \\ (p_j,\ q_j + 1) \mid n \text{ from } O_2 \cup O_3 \\ (p_j - 1,\ q_j) \mid n \text{ from } O_4 \cup O_5 \\ (p_j,\ q_j - 1) \mid n \text{ from } O_6 \cup O_7 \end{cases} \qquad (3.4)$$

Lemma 3.2 [6]

For a set Q of lattice points and a given point c there exists a real circle q, such that $Q = IK(q)$ and the center of q is c if and only if the following conditions hold:

(i) Each element of Q has at least two 8-neighbours, $Q \cap Octant_i$ (i=1, ..., 8) is a digital arc.

(ii) For the points m and n_2 defined by (3.1) and (3.4) the following inequality holds: $d^2(m, c) < d^2(n_2, c)$.

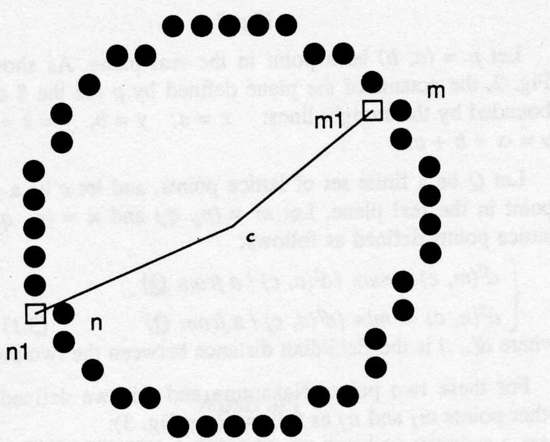


Fig. 3

Section 4 of [1] describes an interesting algorithm for computing the center c of the real circle, which is used in testing condition (ii) of lemma 3.1. But the searches for point c have *exponential time complexity*.

4. Digital-inscribable Quadrilaterals and Digital Circles

Inscribable quadrilaterals (quadrilaterals whose vertices are on a circle) have many interesting properties. Four of them are mentioned here:

Proposition 4.1

A quadrilateral is inscribable if and only if each pair of opposite angles has the sum 180^0 (see Fig. 4a).

Proposition 4.2

A quadrilateral is inscribable if and only if each angle formed by an edge and a diagonal is equal with the angle formed by the opposite edge and the other diagonal of the quadrilateral (see Fig. 4 b).

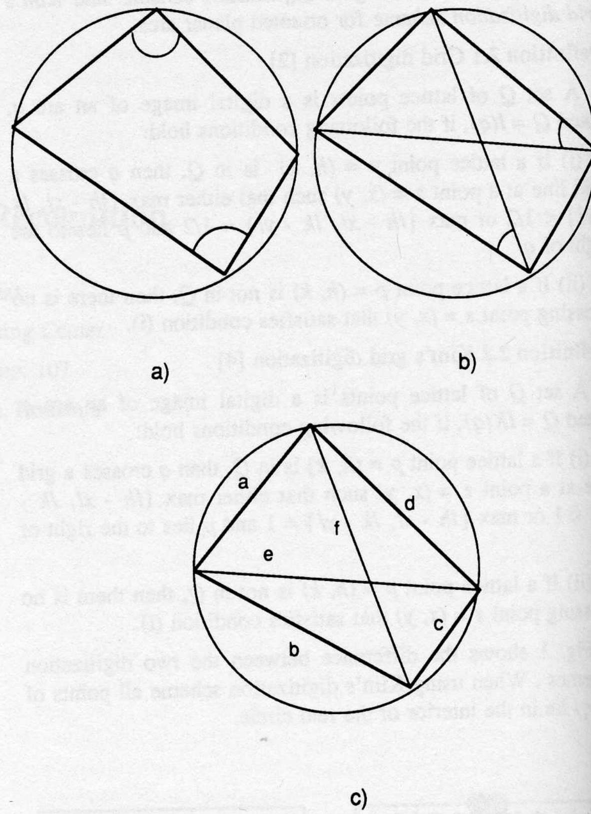


Fig. 4

Proposition 4.3

Let a, b, c, d be the length of the edges of an inscribable quadrilateral, let e and f be the length of its diagonals. Then the following equality holds:

$$f\,e = a\,c + b\,d$$

Proposition 4.4

Let a, b, c, d be the length of the edges of an inscribable quadrilateral, and p its semiperimeter. Then the radius R of the circle that contains the vertices of the quadrilateral can be computed as follows:

$$R^2 = \frac{(a\,b + c\,d)\,(a\,c + b\,d)\,(a\,d + b\,c)}{16\,(p - a)\,(p - b)\,(p - c)\,(p - d)}$$

The proof of this epression is given in appendix.

Definition 4.1

A set of lattice points Q is called *digital-inscribable quadrilateral* if there exists a real-inscribable quadrilateral q such that $I(q) = Q$.

Another important result obtained by Nakamura and Aizawa [7] is an algorithm that computes, for a digital-straight line *s*, and a lattice point *d* *two real-straight lines* $L_n(d)$ *and* $L_m(d)$, *which pass through* *d*, such that all real-straight lines "between" them have the same digital image *s* (see Fig. 5).

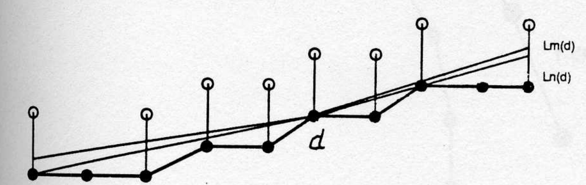


Fig. 5

Consider a digital-inscribable quadrilateral. According to definition 4.1 there exists a real-inscribable quadrilateral *q*, preimage of it. For two consecutive edges f_1 and f_2 we consider the real-straight lines $L_n(d_1)$, $L_m(d_1)$, $L_n(d_2)$, $L_m(d_2)$ (see Fig. 6a) such that all real-straight line segments from the "interval" $[L_n(d_1), L_m(d_1)]$ have the same digital image as f_1 and all real-straight line segments from the "interval" $[L_n(d_2), L_m(d_2)]$ are the "digital equivalent" of f_2.

Let u_a be the angle between $L_n(d_1)$ and $L_m(d_2)$, u_b the angle between $L_n(d_2)$ and $L_m(d_1)$, that is:

$$\begin{cases} u_a = \measuredangle(L_n(d_1), L_m(d_2)) \\ u_b = \measuredangle(L_n(d_2), L_m(d_1)) \end{cases} \quad (4.1)$$

where the angles are considered in the positive interval $[0^0, 180^0]$. With almost the same consideration for the opposite angle of *u* in the quadrilateral, noted *v*, the angles v_a and v_b are defined as follows:

$$\begin{cases} v_a = \measuredangle(L_n(d_4), L_m(d_3)) \\ v_b = \measuredangle(L_n(d_3), L_m(d_4)) \end{cases} \quad (4.2)$$

Lemma 4.1

In a digital-inscribable quadrilateral, for the opposite pairs of angles (u_a, v_a) and (u_b, v_b), defined by (4.1) and (4.2) the following inequalities hold:

$$\begin{cases} ua + va > 180^0 \\ ub + vb < 180^0 \end{cases} \quad (4.3)$$

Proof:

The intersection point between f_1 and f_2 lies on the circle, as f_1 and f_2 are edges of the real-inscribable quadrilateral *q*. Note their angle with *u*. The intersection point between $L_n(d_1)$ (right most of f_1) and $L_m(d_2)$ (left most of f_2) lies inside the circle (see Fig. 6a) and for their angles u_a we obtain:

$$u_a > u \quad (4.4)$$

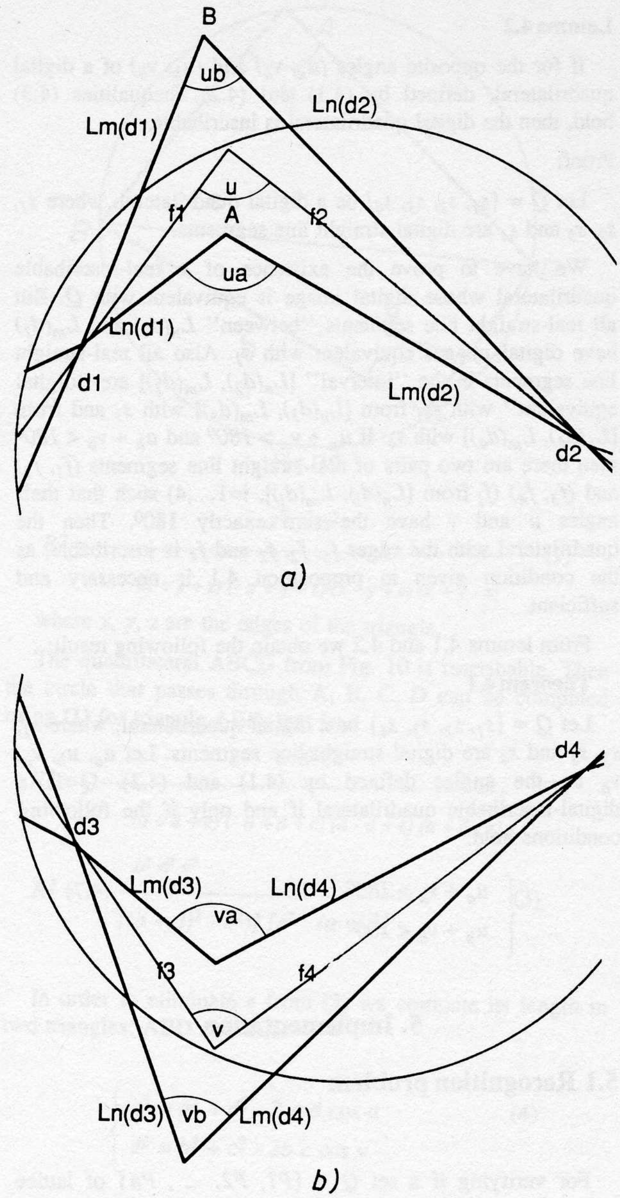


Fig. 6

The intersection point *B* between $L_m(d_1)$ (left most of f_1) and $L_n(d_2)$ (right most of f_2) lies exterior the circle (see Fig. 6a) and for their angle u_b we obtain:

$$u_b < u \quad (4.5)$$

With almost the same consideration for the angles v_a and v_b (see Fig 6b), the following inequalities hold:

$$\begin{cases} v_a > v \\ v_b < v \end{cases} \quad (4.6)$$

The angles are considered in the positive interval $[0^0, 180^0]$; $u + v = 180^0$ as they are opposite angles in an inscribable quadrilateral. So from (4.4), (4.5) and (4.6) we obtain (4.3).

Lemma 4.2

If for the opposite angles (u_a, v_a) and (u_b, v_b) of a digital quadrilateral, defined by (4.1) and (4.2), inequalities (4.3) hold, then the digital quadrilateral is inscribable.

Proof:

Let $Q = \{s_1, s_2, s_3, s_4\}$ be a digital quadrilateral, where s_1, s_2, s_3 and s_4 are digital-straight line segments.

We have to prove the existence of a real-inscribable quadrilateral whose digital image is equivalent with Q. But all real-straight line segments "between" $L_n(d_1)$ and $L_m(d_1)$ have digital images equivalent with s_1. Also all real-straight line segments in the "interval" $[L_n(d_2), L_m(d_2)]$ are "digital equivalent" with s_2, from $[L_n(d_3), L_m(d_3)]$ with s_3 and from $[L_n(d_4), L_m(d_4)]$ with s_4. If $u_a + v_a > 180^0$ and $u_b + v_b < 180^0$ then there are two pairs of real-straight line segments (f_1, f_2) and (f_3, f_4) (f_i from $[L_n(d_i), L_m(d_i)]$, i=1,...,4) such that their angles u and v have the sum exactly 180^0. Then the quadrilateral with the edges f_1, f_2, f_3 and f_4 is inscribable, as the condition given in proposition 4.1 is necessary and sufficient.

From lemma 4.1 and 4.2 we obtain the following result:

Theorem 4.1

Let $Q = \{s_1, s_2, s_3, s_4\}$ be a digital quadrilateral, where s_1, s_2, s_3 and s_4 are digital-straight line segments. Let u_a, u_b, v_a, v_b be the angles defined by (4.1) and (4.2). Q is a digital-inscribable quadrilateral if and only if the following conditions hold:

$$\begin{cases} u_a + v_a > 180^0 \\ u_b + v_b < 180^0 \end{cases} \qquad (4.7)$$

5. Implementation

5.1 Recognition problem

For verifying if a set $Q = \{P1, P2, \ldots, Pn\}$ of lattice points is a digital circle (or arc of a digital circle) we had first, to divide Q into digital-straight line segments. For this purpose was used Sklansky's and Gonzalez's method [11]. This algorithm consists of computing the real-straight line segment l_{ij} between two lattice points, P_i, P_j and the comparison of the distance of all points between P_i and P_j to l_{ij} with a threshold. If there is a point such that its distance to the real-straight line segment l_{ij} is greater than the threshold, then $\{P_i, P_{i+1}, \ldots, P_{j-1}\}$ is a digital-straight line segment. If for all points between P_i and P_j the distance to l_{ij} is less than the threshold, then is tried a "longer" segment, that is $[P_i, P_{j+1}]$. Thresholding is a very difficult problem in pattern recognition. For our tests we admitted 0.1 for a line segment of length 1 unit.

Then we tested for every three consecutive line segments condition (4.3) (see Fig. 7).

We obtained false circles in the case of very "long"

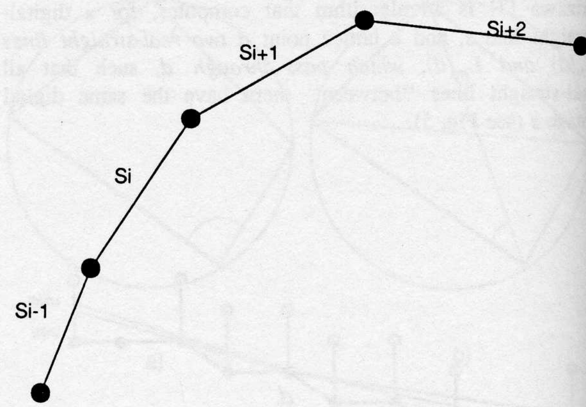


Fig. 7

almost colinear segments, as shown in Fig. 8. In order to eliminate such errors it is necessary an additional test: the distance between the end points of the line segments and the circle must not be greater than a threshold (0.1 for a circle of radius 1 unit).

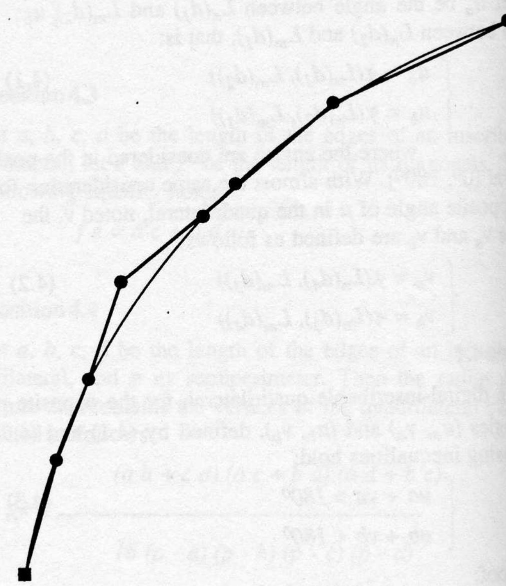


Fig. 8

5.2 Approximation problem

For computing the center and radius of a digital circle we found that the simplest way is to compute, first, the radius, using formula given in proposition 4.4 for all quadrilaterals and then to consider its average value.

5.3 Segmentation problem

In order to find the end points of an arc of a circle, as shown in Fig. 9, is analysed the chain of straight line segments, obtained with Sklansky's and Gonzalez's method and noted $\{S_1, S_2, ..., S_m\}$.

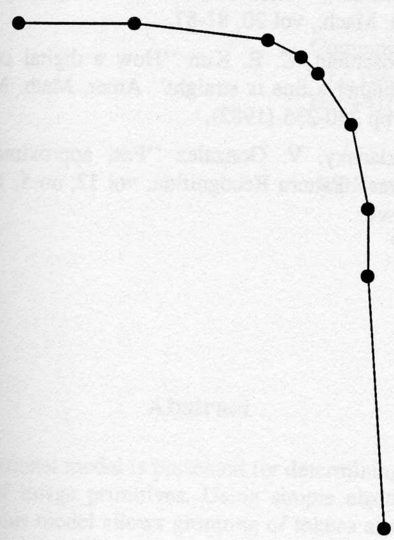


Fig. 9

At least five consecutive line segments, which verify condition (4.3) and distance condition (described in 5.1) were considered necessary for an arc of a digital circle. We advanced in the chain $\{S_1, S_2, ..., S_m\}$ as long as the two mentioned conditions hold, in order to obtain the "longest" arc of a circle.

Conclusions

In section 4 is presented an algorithm that determines whether or not a given set of lattice points is a digital image of a real circle, using digital-inscribable quadrilaterals. These conditions do not need a priori information about the center and the radius of the circle. The set of given lattice points, first must be segmented into digital-straight lines. Then is verified if all three consecutive digital-straight line segments (the fourth edge links the end points of this contour) form a digital-inscribable quadrilateral. In this way is determined if the digital-straight line segments approximate a real circle. This method also permits a simple way to determine the end points of an arc of a digital circle (as described in section 5).

Appendix

The radius R of the circle that passes through the vertices of a triangle, is given by the following formula:

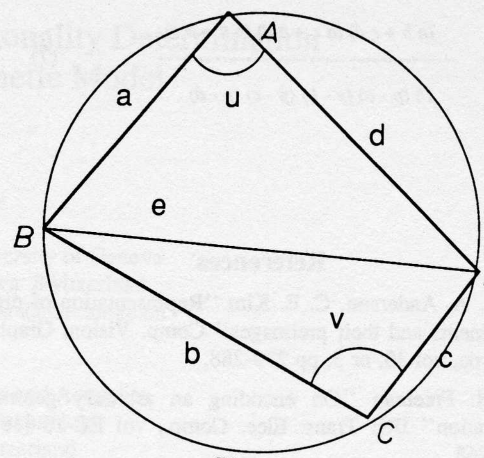


Fig. 10

$$R^2 = \frac{x^2 y^2 z^2}{(x+y+z)(-x+y+z)(x-y+z)(x+y-z)} \quad (1)$$

where x, y, z are the edges of the triangle.

The quadrilateral ABCD from Fig. 10 is inscribable. Then the circle that passes through A, B, C, D can be computed using (1) for triangle ABD, that is:

$$R^2 = \frac{a^2 d^2 e^2}{(a+d+e)(-a+d+e)(a-d+e)(a+d-e)} \quad (2)$$

$$R^2 = \frac{a^2 d^2 e^2}{[(a+d)^2 - e^2)]\,[e^2 - (a-d)^2]} \quad (3)$$

In order to eliminate e from (3) we compute its length in two triangles: ABD and BCD.

$$\begin{cases} e^2 = a^2 + d^2 - 2\,a\,d\cos u \\ e^2 = b^2 + c^2 - 2b\,c\cos v \end{cases} \quad (4)$$

But $\cos u = \cos v$. So from (4) results the following expression of e^2:

$$e^2 = \frac{(b^2 + c^2)\,a\,d + (a^2 + d^2)\,b\,c}{b\,c + a\,d} \quad (5)$$

Using (3) and (5) results the following expresion of R^2:

$$R^2 = \frac{(a\,b + c\,d)\,(a\,c + b\,d)\,(a\,d + b\,c)}{(-a+b+c+d)(a-b+c+d)(a+b-c+d)(a+b+c-d)} \quad (6)$$

Let p be the semiperimeter of the quadrilateral, that is $a+b+c+d=2p$, then:

$$R^2 = \frac{(a b + c d) (a c + b d) (a d + b c)}{16 (p - a) (p - b) (p - c) (p - d)} \qquad (7)$$

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