Attitude Determination by the Support Function and the Curvature Functions

Ying Li
Department of Computer Science
The University of British Columbia
Vancouver, B. C. , Canada, V6T 1Z2

Abstract

This paper proposes to solve the attitude determination problem in robot vision using the support function and the two curvature functions for convex bodies. Mathematical background for these functions is surveyed. The problem of attitude determination is defined as finding a rotation that rotates a measured object to a model. The problem is transformed into constrained optimization problems utilizing the combinations of the support function and each of the two curvature functions. Experiments have been conducted on synthesized shapes. Existing numerical computation packages are employed to solve the optimization problems. The experimental results show that the approach is promising for practical applications.

1 Introduction

This paper proposes to solve the attitude determination problem in robot vision using the support function and the two curvature functions for convex bodies. Attitude determination is a subtask that a robot vision system needs to perform. The support function and the two curvature functions are orientation based representations of shape.

Earlier mathematical results on the support function and the curvature functions have been examined. All mathematical background for this research has been established decades ago (see Bonnesen-Fenchel [1] and Busemann [2]). The problems of attitude determination by the support function and the curvature functions have been formulated as constrained optimization problems for which solutions exist. Experiments have been conducted on synthesized shapes. The experiment results show that the proposed approach is promising for practical applications.

Nalwa [6] proposed the use of the support function in computational vision, but gave no suggestion about how. The Extended Gaussian Image (see Horn [4] and Little [5]) is a special case of a curvature function. It is the second curvature function of a convex polytope. The work presented in this paper deals with dense, instead of sparse, representations of shape.

This paper is organized as follows: Section 2 presents the mathematical background for the support function and the curvature functions. Section 3 defines the problem of attitude determination and proposes an approach to solve the problem using combinations of the support function and each of the two curvature functions. Section 4 describes the experiments on synthesized shapes. Results of attitude determination by the second curvature function and the support function and by the first curvature function and the support function are presented. Section 5 summarizes the present work and provides pointers to problems that we are now working on.

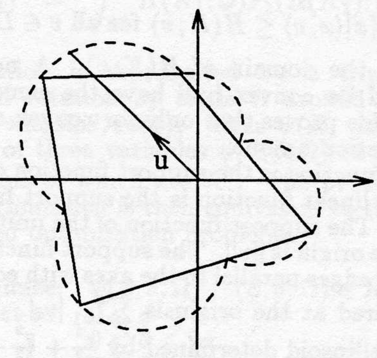


Figure 1: The support function of a convex polygon.

2 Mathematical Background

The support function of a compact convex set K in R^d maps a vector $u \in S^{d-1}$ (S^{d-1} denotes the unit sphere in R^d) to the signed distance between the origin and the support plane of K with outer normal u. Figure 1 shows a convex polygon and its support function. The value of the support function at $u \in S^1$ is the distance between the origin and the dashed arc along the direction determined by u.

Definition 2.1 Let $K \subseteq R^d$ be a nonempty set. The *support function* $H(K; v)$ of K is the real-valued function defined by

$$H(K; v) = \sup \{\langle x, v\rangle \mid x \in K\},$$

for all $v \in R^d$ for which the supremum is finite.

The following properties of the support function show that the support function is a good representation of convex shapes. Support functions are additive, positively homogeneous, and convex. If a set K is rotated, its support function rotates in the same way, i.e.,

$$H(R(K); v) = H(K; R^{-1}(v)) \,. \tag{1}$$

The support function is unique for convex sets, i.e., if K_1, K_2 are nonempty compact convex sets in R^d such that $H(K_1; v) = H(K_2; v)$ for every $v \in R^d$, then $K_1 = K_2$. In fact, a compact convex set K can be represented by its support function as

$$K = \{x | \langle x, v\rangle \leq H(K; v) \text{ for all } v \in D\} \,,$$

where D is the domain of $H(K; v)$. A non-convex polygon and its convex hull have the same support function. This proves that only for convex sets is the support function unique.

A linear function is the support function of a point. A piecewise linear function is the support function of a polytope. The support function of the unit ball with center at the origin is $||v||$. The support function of the cube having edges parallel to the axes with edge length 2 and centered at the origin is $\sum_{i=1}^{d} |v_i|$. Let $E_{a,b,c}$ denote the ellipsoid determined by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. The support function of $E_{a,b,c}$ is

$$H(E_{a,b,c}; v) = \sqrt{a^2 v_1^2 + b^2 v_2^2 + c^2 v_3^2} \,, \tag{2}$$

for $v = (v_1, v_2, v_3) \in R^3$.

Let K be a convex body. Suppose the support function $H(K; v)$ of K has continuous second derivatives. Let (H_{ij}) denote the Hessian matrix of $H(K; v)$, i.e.,

$$(H_{ij}) = \left(\frac{\partial^2 H(K; v)}{\partial v_i \partial v_j} \right) .$$

Define $F_i(K; u)$ to be equal to the sum of all i-rowed principal minors of (H_{ij}) evaluated at u, $i = 1, \ldots, d-1$, and call it the i-th *curvature function* of the convex body K.

For $u \in S^{d-1}$, let $x(K; u)$ be the point on the surface of K with outer normal u. Let $R_1(K; u), R_2(K; u), \cdots, R_{d-1}(K; u)$ be the radii of principal curvature at $x(K; u)$. It is known that

$$\begin{aligned} F_1(K; u) &= R_1(K; u) + \cdots + R_{d-1}(K; u) \,, \\ F_{d-1}(K; u) &= R_1(K; u) R_2(K; u) \cdots R_{d-1}(K; u) \,. \end{aligned}$$

When $d = 3$, there are two curvature functions, F_1 and F_2, which are the sum of radii of principal curvature and the product of radii of principal curvature, respectively. Note that the second curvature function is the reciprocal of the Gaussian curvature, and that the first curvature function is equal to mean curvature divided by Gaussian curvature.

From the support function of ellipsoid $E_{a,b,c}$ the curvature functions of $E_{a,b,c}$ are obtained:

$$F_2(E_{a,b,c}; u) = \frac{a^2 b^2 c^2 (u_1^2 + u_2^2 + u_3^2)}{(a^2 u_1^2 + b^2 u_2^2 + c^2 u_3^2)^2} \,, \tag{3}$$

$$F_1(E_{a,b,c}; u) = \frac{a^2 b^2 (u_1^2 + u_2^2) + b^2 c^2 (u_2^2 + u_3^2) + a^2 c^2 (u_1^2 + u_3^2)}{(a^2 u_1^2 + b^2 u_2^2 + c^2 u_3^2)^{3/2}} \,. \tag{4}$$

The notion of *mixed volume* plays an important role in the studies of convex bodies (see Bonnesen-Fenchel [1] for definition). Let $V(K_1, \ldots, K_d)$ denote the mixed volume of convex bodies $K_1, \ldots, K_d$ in R^d. It does not depend on the order in which K_i appears. The following mixed volume combines the support function and the curvature functions of the convex bodies:

$$V(K_1, \underbrace{K_2, \ldots, K_2}_{i}, \underbrace{B^d, \ldots, B^d}_{d-1-i}) = \frac{1}{d} \binom{d-1}{i}^{-1} \int_{S^{d-1}} H(K_1; \omega) F_i(K_2; \omega) d\omega \,,$$

$i = 1, 2, \ldots, d-1$, where B^d denotes the unit ball in R^d. When $d = 3$, there are two such mixed volumes:

$$V(K_1, K_2, K_2) = \frac{1}{3} \int_{S^{d-1}} H(K_1; \omega) F_2(K_2; \omega) d\omega \,, \tag{5}$$

$$V(K_1, K_2, B^3) = \frac{1}{6} \int_{S^{d-1}} H(K_1; \omega) F_1(K_2; \omega) d\omega \,. \tag{6}$$

Let $V(K)$ denote the volume of K. The following two theorems are fundamental to our solutions to the attitude determination problem.

Theorem 2.1 (Minkowski's Inequality, see page 48 of Busemann [2].)

$$V^d(K_0, K_1, \ldots, K_1) \geq V(K_0) V^{d-1}(K_1) \,.$$

If K_0 and K_1 do not lie in parallel hypersurfaces the equality sign holds only when K_0 and K_1 are homothetic. (Two sets P and Q are said to be homothetic if $P = \{a\} + \lambda Q, a \in R^d, \lambda > 0$.)

Theorem 2.2 (Busemann [2] page 49.)

$$V^2(K_0, K_1, K_1', \ldots, K_{d-2}') \geq V(K_0, K_0, K_1', \ldots, K_{d-2}') \cdot V(K_1, K_1, K_1', \ldots, K_{d-2}') \,,$$

with equality if and only if K_0 and K_1 are homothetic.

3 Problem Definition and Solutions

Generic tasks that robot vision systems perform are [9]: 1) recognition 2) localization and 3) inspection. Attitude determination is a subtask of localization.

Definition 3.1 The attitude determination problem is defined as finding a rotation R such that $R(K') = K$, where K is a model, K' is a measured object that is resulted by an unknown rotation from K.

Let K_1, K_2 be convex bodies in R^3. By Minkowski's Inequality (Theorem 2.1),

$$V^3(K_1, K_2, K_2) \geq V(K_1)V^2(K_2) , \qquad (7)$$

the equality holds if and only if K_1 and K_2 are homothetic. By Theorem 2.2,

$$V^2(K_1, K_2, B^3) \geq V(K_1, K_1, B^3)V(K_2, K_2, B^3) , \quad (8)$$

the equality holds if and only if K_1 and K_2 are homothetic. It is known that $d \cdot V(K, \ldots, K, B^d)$ is equal to the surface area $S(K)$ of K. Combining equations (5) and (6) with inequalities (7) and (8), the following conclusions are obtained.

1. Among all convex bodies K_1 of volume 1, those that are homothetic to K_2 yield the minimal value of
$$\int_{S^2} H(K_1;\omega)F_2(K_2;\omega)d\omega \; ;$$
2. Among all convex bodies K_1 of surface area 1, those that are homothetic to K_2 yield the minimal value of
$$\int_{S^2} H(K_1;\omega)F_1(K_2;\omega)d\omega \; .$$

Let K_1 and K_2 be two convex bodies that have the same volume or the same surface area. Define functions of rotation R:

$$\psi(R) \triangleq \int_{S^2} H(R^{-1}(K_1);\omega)F_2(K_2;\omega)d\omega , \quad (9)$$

$$\varphi(R) \triangleq \int_{S^2} H(R^{-1}(K_1);\omega)F_1(K_2;\omega)d\omega . \quad (10)$$

Since rotations preserve volumes and surface areas, functions $\psi(R)$ and $\varphi(R)$ reach minima when and only when $R^{-1}(K_1) = K_2$, i.e., $R(K_2) = K_1$. By Definition 3.1, the problem of attitude determination can be solved if and only if the minima of $\psi(R)$ or $\varphi(R)$ can be found, and the minima are solutions to the attitude determination problem. Recall (7), (8), and the definition of $\psi(R)$ and $\varphi(R)$, the global minimum of $\psi(R)$ is $3\sqrt[3]{V^2(K_2)V(K_1)}$, and that of $\varphi(R)$ is $2\sqrt{S(K_1)S(K_2)}$. By the if-and-only-if condition in Theorem 2.1 and Theorem 2.2, the global minima are unique, modulo any rotational symmetries that K_1 possesses. Thus the attitude determination problem can be expressed as one of the following optimization problems:

$$\begin{array}{ll} \text{minimize:} & \psi(R) \\ \text{subject to:} & R \text{ is a rotation} , \end{array} \qquad (11)$$

or

$$\begin{array}{ll} \text{minimize:} & \varphi(R) \\ \text{subject to:} & R \text{ is a rotation} . \end{array} \qquad (12)$$

Equivalently, the attitude determination problem may also be expressed as minimizing $\psi_1(R)$, $\varphi_1(R)$, $\psi_2(R)$, or $\varphi_2(R)$, which are defined as:

$$\begin{aligned} \psi_1(R) &\triangleq \int_{S^2} H(R(K_2);\omega)F_2(K_1;\omega)d\omega , \\ \varphi_1(R) &\triangleq \int_{S^2} H(R(K_2);\omega)F_1(K_1;\omega)d\omega , \\ \psi_2(R) &\triangleq \int_{S^2} H(K_1;\omega)F_2(R(K_2);\omega)d\omega , \\ \varphi_2(R) &\triangleq \int_{S^2} H(K_1;\omega)F_1(R(K_2);\omega)d\omega . \end{aligned}$$

A rotation is represented by a triple (ϕ, θ, Ω) meaning a rotation by angle Ω around unit vector $(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$. Thus $\psi(R)$ and $\varphi(R)$ are functions of three variables $\psi(\phi,\theta,\Omega)$ and $\varphi(\phi,\theta,\Omega)$, the domains of which are R^3. The problem of attitude determination is then equivalent to the following constrained optimization problems:

$$\begin{array}{ll} \text{minimize:} & \psi(\phi,\theta,\Omega) , \ (\phi,\theta,\Omega) \in R^3 , \\ \text{subject to:} & 0 \leq \phi \leq \pi , \\ & 0 \leq \theta \leq 2\pi , \\ & 0 \leq \Omega \leq \pi , \end{array} \qquad (13)$$

or

$$\begin{array}{ll} \text{minimize:} & \varphi(\phi,\theta,\Omega) , \ (\phi,\theta,\Omega) \in R^3 , \\ \text{subject to:} & 0 \leq \phi \leq \pi , \\ & 0 \leq \theta \leq 2\pi , \\ & 0 \leq \Omega \leq \pi . \end{array} \qquad (14)$$

Since the feasible regions of the optimization problems (13) and (14) are closed and bounded, solutions to the optimization problems exist.

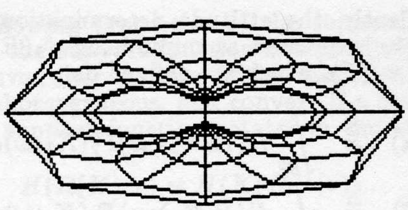
4 Experiments on Synthesized Shapes

Experiments have been conducted on synthesized shapes to solve the attitude determination problem following the approach proposed in Section 3. The synthesized convex surfaces have been the ellipsoid $E_{3,5,9}$ so far. The support function and the curvature functions of $E_{3,5,9}$ are

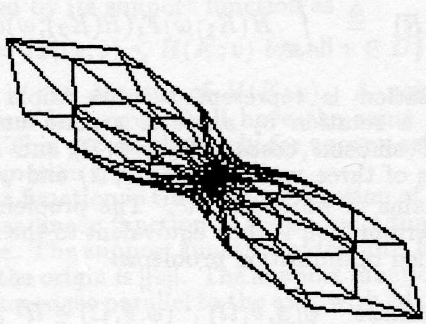
$$\begin{aligned} H(E_{3,5,9};v) &= \sqrt{9v_1^2 + 25v_2^2 + 81v_3^2} , \\ F_2(E_{3,5,9};v) &= \frac{18225(v_1^2 + v_2^2 + v_3^2)}{(9v_1^2 + 25v_2^2 + 81v_3^2)^2} , \\ F_1(E_{3,5,9};v) &= \frac{954v_1^2 + 2250v_2^2 + 2754v_3^2}{(9v_1^2 + 25v_2^2 + 81v_3^2)^{3/2}} . \end{aligned}$$

Figure 2, Figure 3, and Figure 4 draw the second curvature function, the first curvature function, and the support function of $E_{3,5,9}$, respectively.

Let R_0 be a fixed rotation, $K_1 \triangleq R_0(E_{3,5,9})$. Think of K_1 as a model, and $K_2 \triangleq R_0^{-1}(K_1) = E_{3,5,9}$ as an object for which certain measurements have been

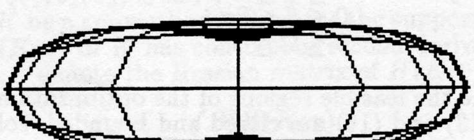


(a)



(b)

Figure 2: The second curvature functions of $E_{3,5,9}$. (a): orthogonally projected onto yz−plane; (b): orthogonally projected onto a plane perpendicular to (1, 1, 1).



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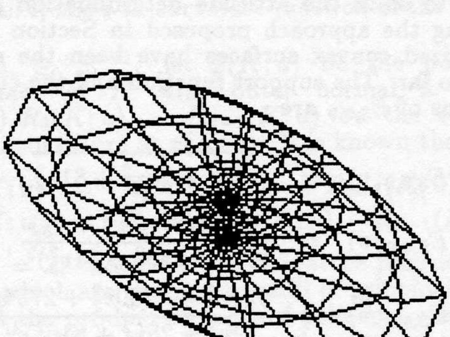
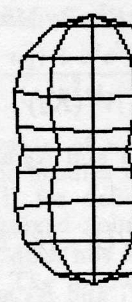
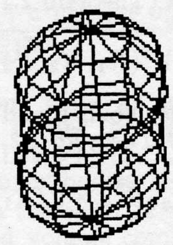


(b)

Figure 3: The first curvature functions of $E_{3,5,9}$. (a): orthogonally projected onto yz−plane; (b): orthogonally projected onto a plane perpendicular to (1, 1, 1).





(a) (b)

Figure 4: The support function of $E_{3,5,9}$. (a): orthogonally projected onto yz−plane; (b): orthogonally projected onto a plane perpendicular to (1, 1, 1).

made. The goal of the experiments is to find R_0 through the support functions and the curvature functions of K_1 and K_2. By (1), the objective functions for the special case of $E_{3,5,9}$ are

$$\psi(R) = \int_{S^2} H(E_{3,5,9}; R_0^{-1}(R(\omega)))F_2(E_{3,5,9};\omega)d\omega \, ,$$

$$\varphi(R) = \int_{S^2} H(E_{3,5,9}; R_0^{-1}(R(\omega)))F_1(E_{3,5,9};\omega)d\omega \, ,$$

$$\psi_1(R) = \int_{S^2} H(E_{3,5,9}; R^{-1}(\omega))F_2(E_{3,5,9}; R_0^{-1}(\omega))d\omega \, ,$$

$$\varphi_1(R) = \int_{S^2} H(E_{3,5,9}; R^{-1}(\omega))F_1(E_{3,5,9}; R_0^{-1}(\omega))d\omega \, .$$

$$\psi_2(R) = \int_{S^2} H(E_{3,5,9}; R_0^{-1}(\omega))F_2(E_{3,5,9}; R^{-1}(\omega))d\omega \, ,$$

$$\varphi_2(R) = \int_{S^2} H(E_{3,5,9}; R_0^{-1}(\omega))F_1(E_{3,5,9}; R^{-1}(\omega))d\omega \, .$$

The R_0 for the experiments is $(\pi/6, \pi/9, \pi/4)$.

The roles of the model and the object are interchangeable depending on what measurement can be made about the object. For example, suppose the support function of an object can be measured instead of curvature functions. Then the role of K_1 and K_2 in the above process can be exchanged, and we need to know the curvature functions, instead of the support function, of the model.

A constrained nonlinear programming subroutine, NLPQL [8], has been used to solve the optimization problems (13) and (14). A numerical integration subroutine from [3] is used to calculate the objective functions. A surface integral $\int_{S^2} f(\omega)d\omega$ is transformed into the volume integral $\int_0^{\pi}\int_0^{2\pi} f(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)\sin\phi d\theta d\phi$.

$\psi(R)$	$\varphi(R)$	$\psi_1(R)$	$\varphi_1(R)$	$\psi_2(R)$	$\varphi_2(R)$
1070	1180	859	1178	982	1163

Table 1: Results of experiments: the number of initial guesses out of 4096 initial guesses, from which NLPQL finds the optimal point R_0.

The optimization subroutine employed does not guarantee, like many optimization packages, that the minima found are global. Since we know the global minima of $\psi(R)$ and $\varphi(R)$, we can always check whether a minimum found by the subroutine is the global minimum.

Optimization results have been obtained when all functions are given by analytical expressions. The optimization processes have been executed 4096 times, each time with a different input point as an initial guess of the optimal point. These points are $(0.1 \times i, 0.2 \times j, 0.1 \times k), i, j, k = 1, 3, 5, \ldots, 31$. The triples (ϕ, θ, Ω) obtained fall into two categories: 1. the rotation R_0; 2. a boundary point of the feasible region. The numbers listed in Table 1 indicate the numbers of initial guesses from which the global optimal point, R_0, is found by the optimization processes. The distribution of the two categories over the initial guesses suggests connected regions within the feasible region, which may assist selecting favorable initial guesses to obtain R_0. The accuracy of the solutions can be controlled by the optimization subroutine.

As examples, the results of the optimization processes on $\psi(R)$ and $\varphi(R)$ with initial guess $(0.4, 1.05, 0.15)$ are shown in Figure 5 and Figure 6, respectively. In the figures, the shapes in thick gray are the models and the shapes in solid black are the shapes obtained at different iterations.

Similar optimization results have also been obtained when the curvature functions are given by sampled data. Suppose $F_i(E_{3,5,9};\xi)$ is given by a set of discrete samples, i.e., only at a finite set of points in S^2 is the function value of $F_i(E_{3,5,9};\xi)$ given. Using an interpolating scheme [7] an estimate of $F_i(E_{3,5,9};\xi)$ at any point $\xi \in S^2$ can be obtained. Denote the interpolated function by $F_i^s(E_{3,5,9};\xi)$. The functions to be minimized are

$$\psi(R) = \int_{S^2} H(E_{3,5,9}; R_0^{-1}(R(\omega)))F_2^s(E_{3,5,9};\omega)d\omega ,$$

$$\varphi(R) = \int_{S^2} H(E_{3,5,9}; R_0^{-1}(R(\omega)))F_1^s(E_{3,5,9};\omega)d\omega .$$

For $\psi(R)$, the optimization process was run with 75 different initial guesses, 44 of which lead to the optimal point R_0. For $\varphi(R)$, 91 initial guesses were tried and 68 of them resulted in the optimal point R_0.

Similar experiments on $\psi_1(R)$, $\varphi_1(R)$, $\psi_2(R)$, and $\varphi_2(R)$ are being conducted at the time of this writing.

5 Summary and Future Research

An approach to solve the problem of attitude determination by the support function and the curvature functions has been proposed. The problem of attitude determination is transformed into constrained optimization problems by the aid of existing mathematical results. Solutions to the optimization problems have been proved to exist. Experiments have been conducted on synthesized shapes both when the support functions and the curvature functions are given by analytical expressions and when the support function is given analytically and the curvature functions are given by sampled data. Good matching results have been obtained.

The next step is to experiment under the condition that all functions are given by sampled data. After that, experiments on real shapes obtained through visual equipment are expected to be conducted. It is also worthwhile to compare the effect of the first curvature function on the optimization process with that of the second curvature function.

Although it can be determined whether a solution found by the optimization routine is the global minimum, it would be nice to be able to answer the question of whether it is possible to find the solution that achieves global minimum within a fixed number of trials of initial guesses. If yes, what is the minimal number of trials. Theoretical investigations on the rotation space are needed in order to answer these questions.

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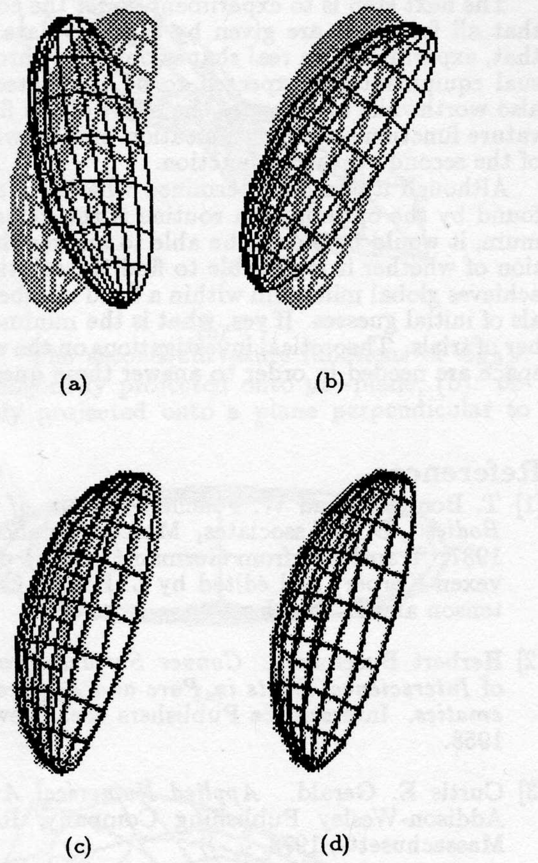

Figure 5: One matching process for $E_{3,5,9}$ using $\psi(R)$, the combination of the support function and the second curvature function. Rotation R_0 is $(\pi/6, \pi/9, \pi/4)$. Initial guess is $(0.4, 1.45, 1.15)$. From (a) to (d): 0th, 3rd, 6th, and the final matching result.

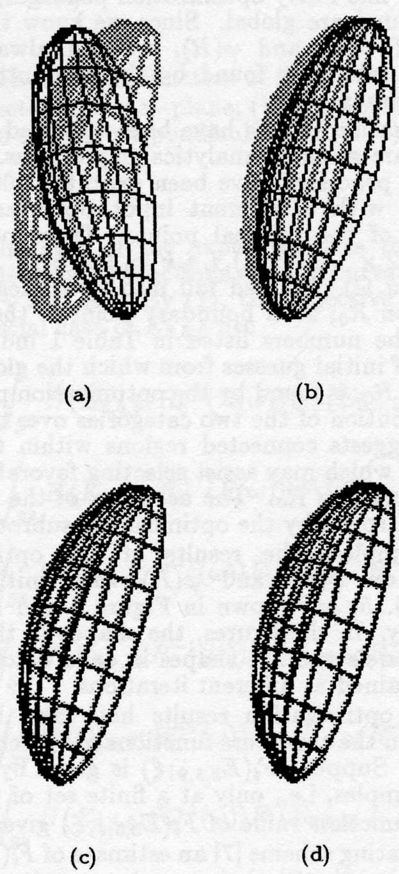

Figure 6: One matching process for $E_{3,5,9}$ using $\varphi(R)$, the combination of the support function and the first curvature function. Rotation R_0 is $(\pi/6, \pi/9, \pi/4)$. Initial guess is $(0.4, 1.45, 1.15)$. From (a) to (d): 0th, 3rd, 6th, and the final matching result.