The Unscented Kalman Filter for State Estimation

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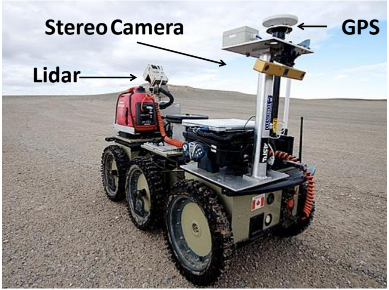
Outline

Defining terms



State:

$$p(\mathbf{x}_k | \mathbf{u}_{1:k}, \mathbf{y}_{1:k}) \rightarrow \mathcal{N}\left(\hat{\mathbf{x}}_k, \hat{\mathbf{P}}_k\right)$$

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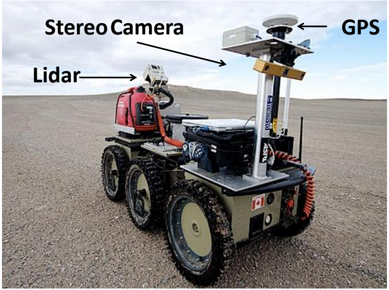

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$$\mathbf{x}_k = \mathbf{h}\left(\mathbf{x}_{k-1}, \mathbf{u}_k, \mathbf{w}_k\right), \quad \mathbf{w}_k \sim \mathcal{N}\left(\mathbf{0}, \mathbf{Q}_k\right)$$

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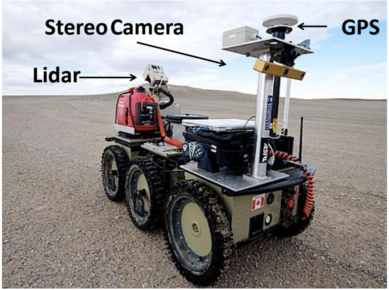

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Sensor model:

$$\mathbf{y}_k = \mathbf{g}\left(\mathbf{x}_k,\mathbf{n}_k\right), \quad \mathbf{n}_k \sim \mathcal{N}\left(\mathbf{0},\mathbf{R}_k\right)$$

Goal

Goal at time-step k:

$$\{\hat{\mathbf{x}}_{k-1}, \hat{\mathbf{P}}_{k-1}, \mathbf{u}_k, \mathbf{y}_k\} \rightarrow \{\hat{\mathbf{x}}_k, \hat{\mathbf{P}}_k\}$$

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Measurement update (correction step):

$$\begin{aligned} \hat{\mathbf{x}}_k &= \hat{\mathbf{x}}_k^- + \mathbf{K}_k\left(\mathbf{y}_k - \hat{\mathbf{y}}_k\right) \\ \hat{\mathbf{P}}_k &= \hat{\mathbf{P}}_k^- - \mathbf{K}_k \mathbf{U}_k^T \end{aligned}$$

- $\hat{\mathbf{x}}_k^-$: Predicted state
- $\hat{\mathbf{P}}_k^-$: Predicted covariance
- $\mathbf{K}_k$: Kalman gain
- $\hat{\mathbf{y}}_k$: Predicted measurement
- $\mathbf{U}_k$: Cross-covariance term

Extended Kalman Filter

- Nonlinear extension to the famous 'Kalman Filter'
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- Prediction step:

$$\begin{aligned} \hat{\mathbf{x}}_k^- &= \mathbf{h}(\hat{\mathbf{x}}_{k-1}, \mathbf{u}_k, \mathbf{0}) \\ \hat{\mathbf{P}}_k^- &= \mathbf{H}_{\mathbf{x},k} \hat{\mathbf{P}}_{k-1} \mathbf{H}_{\mathbf{x},k}^T + \mathbf{H}_{\mathbf{w},k} \mathbf{Q}_k \mathbf{H}_{\mathbf{w},k}^T \end{aligned}$$

where

$$\mathbf{H}_{\mathbf{x},k} := \left. \frac{\partial \mathbf{h}(\mathbf{x}_{k-1}, \mathbf{u}_k, \mathbf{w}_k)}{\partial \mathbf{x}_{k-1}} \right|_{\hat{\mathbf{x}}_{k-1}, \mathbf{u}_k, \mathbf{0}}, \quad \mathbf{H}_{\mathbf{w},k} := \left. \frac{\partial \mathbf{h}(\mathbf{x}_{k-1}, \mathbf{u}_k, \mathbf{w}_k)}{\partial \mathbf{w}_k} \right|_{\hat{\mathbf{x}}_{k-1}, \mathbf{u}_k, \mathbf{0}}.$$

Example

- Consider a simple example with a quadratic nonlinearity,

$$h\left(x_{k-1}, u_k, w_k\right) = x_{k-1}^2 + w_k$$

Example

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Strengths and weaknesses

Strengths:

- Computationally inexpensive

Weaknesses:

- For highly nonlinear problems, the EKF is known to encounter issues with both accuracy and stability (Julier et al. ,1995; Wan and van der Merwe, 2000)
- When analytical Jacobians are not available, numerical Jacobians are required

Unscented Kalman Filter

- Introduced by Julier et al. (1995) as a derivative-free alternative to the EKF
- UKF uses a weighted set of deterministically sampled points called *sigma-points*, which are passed through the nonlinearity and are used to approximate the statistics of the distribution
- Unscented Transformation is accurate to at least third-order for Gaussian systems

Unscented Kalman Filter

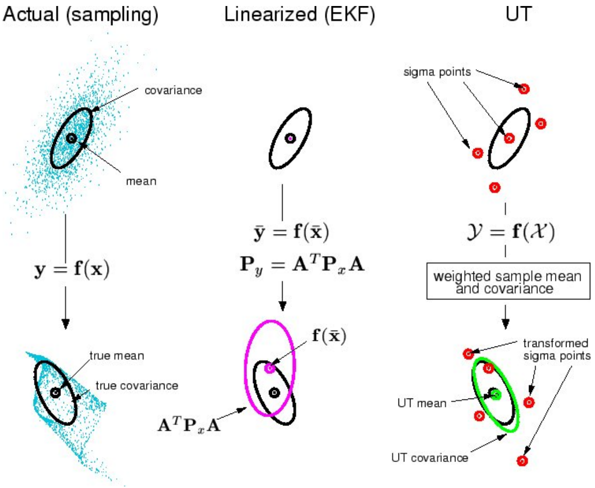

Image taken from van der Merwe and Wan (2001).

The Unscented Transformation

A set of $2N+1$ sigma-points is computed from the prior density, $\mathcal{N}\left(\hat{\mathbf{x}}_{k-1}, \hat{\mathbf{P}}_{k-1}\right)$, according to

$$\begin{aligned}
\mathbf{S}\mathbf{S}^T &:= \hat{\mathbf{P}}_{k-1} \quad \text{(Cholesky decomposition)} \\
\mathcal{X}_0 &:= \hat{\mathbf{x}}_{k-1} \\
\mathcal{X}_i &:= \hat{\mathbf{x}}_{k-1} + \sqrt{N+\kappa}\,\text{col}_i\mathbf{S} \\
\mathcal{X}_{i+N} &:= \hat{\mathbf{x}}_{k-1} - \sqrt{N+\kappa}\,\text{col}_i\mathbf{S}
\end{aligned} \qquad i = 1 \ldots N$$

where $N = \text{dim}(\hat{\mathbf{x}}_{k-1})$.

$$\mathcal{X}_i^- = \mathbf{h}\left(\mathcal{X}_i\right), \quad \text{for } i = 0 \ldots 2N$$

The Unscented Transformation

The mean and covariance are computed as follows:

$$\hat{\mathbf{x}}_k^- = \frac{1}{N+\kappa}\left(\kappa \mathcal{X}_0^- + \frac{1}{2}\sum_{i=1}^{2N} \mathcal{X}_i^-\right),$$

$$\hat{\mathbf{P}}_k^- = \frac{1}{N+\kappa}\left(\kappa\left(\mathcal{X}_0^- - \hat{\mathbf{x}}\right)\left(\mathcal{X}_0^- - \hat{\mathbf{x}}\right)^T + \frac{1}{2}\sum_{i=1}^{2N}\left(\mathcal{X}_i^- - \hat{\mathbf{x}}\right)\left(\mathcal{X}_i^- - \hat{\mathbf{x}}\right)^T\right).$$

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- Why this deterministic sampling scheme?

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- Why this deterministic sampling scheme?
- Consider the expectation of the prior mean plus a disturbance:

$$\hat{\mathbf{x}}_k^- = \mathsf{E}\left[\mathbf{h}\left(\hat{\mathbf{x}}_{k-1} + \Delta\mathbf{x}\right)\right], \quad \Delta\mathbf{x} \sim \mathcal{N}\left(\mathbf{0}, \hat{\mathbf{P}}_{k-1}\right)$$

Analytical Linearization

$$\begin{aligned}\hat{\mathbf{x}}_k^- &= \mathsf{E}\left[\mathbf{h}\left(\hat{\mathbf{x}}_{k-1} + \Delta\mathbf{x}\right)\right] \\ &= \mathbf{h}\left(\hat{\mathbf{x}}_{k-1}\right) + \mathsf{E}\left[\mathbf{D}_{\Delta\mathbf{x}}\mathbf{h} + \frac{\mathbf{D}^2_{\Delta\mathbf{x}}\mathbf{h}}{2!} + \frac{\mathbf{D}^3_{\Delta\mathbf{x}}\mathbf{h}}{3!} + \frac{\mathbf{D}^4_{\Delta\mathbf{x}}\mathbf{h}}{4!} + \dots\right]\end{aligned}$$

where,

$$\frac{\mathbf{D}^n_{\Delta\mathbf{x}}\mathbf{h}}{n!} = \frac{1}{n!}\left(\sum_{i=1}^{N} \Delta x_n \frac{\partial}{\partial x_n}\right)^n \mathbf{h}\left(\mathbf{x}\right)\Bigg|_{\mathbf{x}=\hat{\mathbf{x}}_{k-1}}$$

Analytical Linearization

By symmetry, odd-moments are zero,

$$\hat{\mathbf{x}}_k^- = \mathbf{h}\left(\hat{\mathbf{x}}_{k-1}\right) + \mathsf{E}\left[\frac{\mathbf{D}^2_{\Delta\mathbf{x}}\mathbf{h}}{2!} + \frac{\mathbf{D}^4_{\Delta\mathbf{x}}\mathbf{h}}{4!} + \dots\right]$$

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The second-order term is given by,

$$\mathsf{E}\left[\frac{\mathbf{D}_{\Delta\mathbf{x}}^2\mathbf{h}}{2!}\right] = \left(\frac{\nabla^T\mathsf{E}\left[\Delta\mathbf{x}\Delta\mathbf{x}^T\right]\nabla}{2!}\right)\mathbf{h}\Bigg|_{\mathbf{x}=\hat{\mathbf{x}}_{k-1}} = \left(\frac{\nabla^T\hat{\mathbf{P}}_{k-1}\nabla}{2!}\right)\mathbf{h}\Bigg|_{\mathbf{x}=\hat{\mathbf{x}}_{k-1}}$$

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Resulting in the following

$$\hat{\mathbf{x}}_k^- = \mathbf{h}\left(\hat{\mathbf{x}}_{k-1}\right) + \left(\frac{\nabla^T\hat{\mathbf{P}}_{k-1}\nabla}{2!}\right)\mathbf{h}\Big|_{\mathbf{x}=\hat{\mathbf{x}}_{k-1}} + \mathsf{E}\left[\frac{\mathbf{D}^4_{\Delta\mathbf{x}}\mathbf{h}}{4!} + \dots\right]$$

Statistical Linearization

We define some samples according to

$$\begin{aligned} \mathcal{X}_i &:= \hat{\mathbf{x}}_{k-1} + \sigma_i \\ \mathcal{X}_i^- &= \mathbf{h}(\mathcal{X}_i) \end{aligned} \qquad i = 0 \ldots 2N$$

where $N = \dim(\hat{\mathbf{x}}_{k-1})$. Now we calculate the weighted sigma-point expectations and compare with the 'true' mean

Statistical Linearization

$$\hat{\mathbf{x}}_k^- = \mathbf{h}\left(\hat{\mathbf{x}}_{k-1}\right) + \frac{1}{2(N+\kappa)} \sum_{i=1}^{2N} \left(\mathbf{D}_{\sigma_i}\mathbf{h} + \frac{\mathbf{D}_{\sigma_i}^2\mathbf{h}}{2!} + \frac{\mathbf{D}_{\sigma_i}^3\mathbf{h}}{3!} + \frac{\mathbf{D}_{\sigma_i}^4\mathbf{h}}{4!} + \dots \right)$$

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The second-order term is given by

$$\frac{\mathbf{D}_{\sigma_i}^2\mathbf{h}}{2!} = \left(\frac{\nabla^T \sigma_i \sigma_i^T \nabla}{2!} \right) \mathbf{h} \Big|_{\mathbf{x}=\hat{\mathbf{x}}_{k-1}}$$

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$$\hat{\mathbf{x}}_k^- = \mathbf{h}\left(\hat{\mathbf{x}}_{k-1}\right) + \frac{1}{2(N+\kappa)}\sum_{i=1}^{2N}\left(\mathbf{D}_{\sigma_i}\mathbf{h} + \frac{\mathbf{D}_{\sigma_i}^2\mathbf{h}}{2!} + \frac{\mathbf{D}_{\sigma_i}^3\mathbf{h}}{3!} + \frac{\mathbf{D}_{\sigma_i}^4\mathbf{h}}{4!} + \dots\right)$$

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Recall

$$\mathsf{E}\left[\frac{\mathbf{D}_{\Delta\mathbf{x}}^2\mathbf{h}}{2!}\right] = \left(\frac{\nabla^T\mathsf{E}\left[\Delta\mathbf{x}\Delta\mathbf{x}^T\right]\nabla}{2!}\right)\mathbf{h}\Big|_{\mathbf{x}=\hat{\mathbf{x}}_{k-1}} = \left(\frac{\nabla^T\hat{\mathbf{P}}_{k-1}\nabla}{2!}\right)\mathbf{h}\Big|_{\mathbf{x}=\hat{\mathbf{x}}_{k-1}}$$

Statistical Linearization

We let $\sigma_i := \pm\sqrt{N+\kappa}\text{col}_i\mathbf{S}$, where $\mathbf{S}\mathbf{S}^T = \hat{\mathbf{P}}_{k-1}$, which gives

$$\hat{\mathbf{x}}_k^- = \mathbf{h}\left(\hat{\mathbf{x}}_{k-1}\right) + \left(\frac{\nabla^T\hat{\mathbf{P}}_{k-1}\nabla}{2!}\right)\mathbf{h}\Big|_{\mathbf{x}=\hat{\mathbf{x}}_{k-1}} + \frac{1}{2(N+\kappa)}\sum_{i=1}^{2N}\left(\frac{\mathbf{D}_{\sigma_i}^4\mathbf{h}}{4!} + \dots\right)$$

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Compare above with analytical linearization

$$\hat{\mathbf{x}}_k^- = \mathbf{h}\left(\hat{\mathbf{x}}_{k-1}\right) + \left(\frac{\nabla^T \hat{\mathbf{P}}_{k-1} \nabla}{2!}\right) \mathbf{h}\bigg|_{\mathbf{x}=\hat{\mathbf{x}}_{k-1}} + \mathsf{E}\left[\frac{\mathbf{D}_{\Delta\mathbf{x}}^4 \mathbf{h}}{4!} + \dots \right]$$

Example

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Augmented UKF

$$\mathbf{x}_k = \mathbf{h}\left(\mathbf{x}_{k-1}, \mathbf{u}_k, \mathbf{w}_k\right), \qquad \mathbf{w}_k \sim \mathcal{N}\left(\mathbf{0}, \mathbf{Q}_k\right)$$
$$\mathbf{y}_k = \mathbf{g}\left(\mathbf{x}_k, \mathbf{n}_k\right), \qquad \mathbf{n}_k \sim \mathcal{N}\left(\mathbf{0}, \mathbf{R}_k\right)$$

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- Completely general case, both the prior belief, the process noise and measurement noise have uncertainty so these are stacked together in the following way:

$$\mathbf{z} := \begin{bmatrix} \hat{\mathbf{x}}_{k-1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \qquad \mathbf{Y} := \begin{bmatrix} \hat{\mathbf{P}}_{k-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_k \end{bmatrix}$$

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- Let $N := \dim\left(\hat{\mathbf{x}}\right)$, $P := \dim\left(\mathbf{Q}_k\right)$, $M := \dim\left(\mathbf{R}_k\right)$
- Number of sigma-points: $2(N + P + M) + 1$

Additive noise

- If our motion and observation model are given by

$$\begin{aligned} \mathbf{x}_k &= \mathbf{h}\left(\mathbf{x}_{k-1}, \mathbf{u}_k\right) + \mathbf{w}_k, \quad & \mathbf{w}_k \sim \mathcal{N}\left(\mathbf{0}, \mathbf{Q}_k\right) \\ \mathbf{y}_k &= \mathbf{g}\left(\mathbf{x}_k\right) + \mathbf{n}_k, & \mathbf{n}_k \sim \mathcal{N}\left(\mathbf{0}, \mathbf{R}_k\right) \end{aligned}$$

- Only $2N+1$ sigma-points are required!

$$\mathbf{z} := \hat{\mathbf{x}}_{k-1}, \qquad \mathbf{Y} := \hat{\mathbf{P}}_{k-1}$$

- Must re-draw sigma-points from $\{\hat{\mathbf{x}}_k^-, \hat{\mathbf{P}}_k^-\}$ for the correction step (lose odd-moment information)
- Alternatively, we can use $2(N+P)+1$ sigma-points for the correction step and avoid re-drawing (incorporate $\sqrt{\mathbf{Q}_k}$ into sigma-points)

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Other efficiency improvements

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  - Used a set of $N + 2$ sigma-points, which were chosen to minimize the third-order moments (accurate to second-order)

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- Reduced sigma-point UKF (Quine, 2006)
  - Used a minimal set of $N + 1$ sigma-points (accurate to second-order)
- Square-root Unscented Kalman Filter (van der Merwe and Wan, 2001)
  - Efficient square-root form that avoids refactorizing the state covariance at the prediction step and reduces computational cost required to compute the Kalman gain
  - Present additional benefits in terms of numerical stability and ensures that the state covariance is always positive-definite

Strengths and weaknesses

Strengths:

- Unscented Transformation is accurate to third-order for Gaussian systems
- Derivative-free (easy implementation)

Weaknesses:

- Depending on noise assumptions, can be expensive
- Sigma-point scaling issues

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  - The Scaled Unscented Kalman Filter (Julier, 2002)

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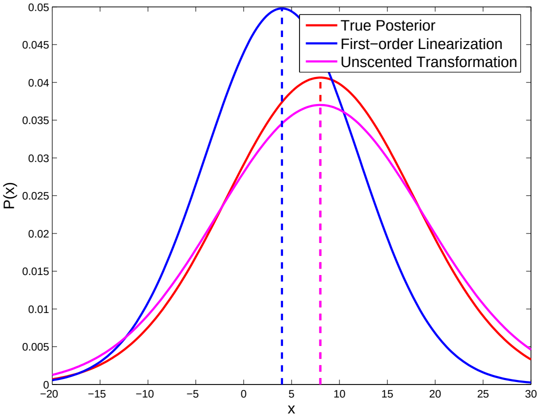
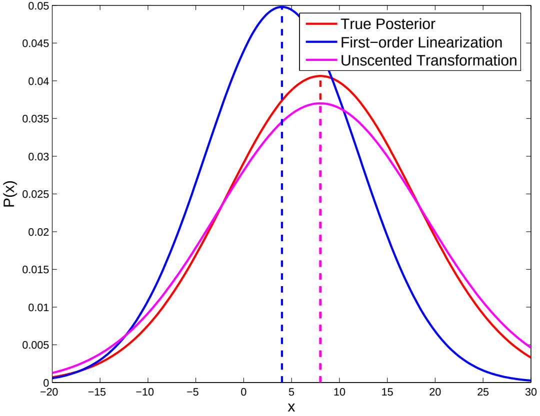
  - The Scaled Unscented Kalman Filter (Julier, 2002)
  - Additional weight parameters (Wan and van der Merwe, 2000)

Accuracy


True Posterior
First−order Linearization
Unscented Transformation
P(x)
X
0.05
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
−20
−15
−10
−5
0
5
10
15
20
25
30

Accuracy



- Tong, C. and Barfoot, T.D. “A Comparison of the EKF, SPKF, and the Bayes Filter for Landmark-Based Localization,” In Proceedings of the 7th Canadian Conference on Computer and Robot Vision (CRV), to appear. Ottawa, Canada, 31 May - 2 June 2010.

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- Sigma point reduction:
  - Spherical-simplex sigma-points (Julier et al., 2003)
  - Reduced sigma-point UKF (Quine, 2006)

Any Questions?

References

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