

# Computing Axis of Symmetry Using Constrained High Curvature Points Matching

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## Abstract

*Finding the axis of symmetry of two curves is posed as a correspondence problem between points of significant curvature along these curves. First, a novel and simple algorithm for computing points of significant curvature is developed. Its main strength is that it does not compute curvature values as such, but computes a robust indication of them. The detected points are then fed to another algorithm which integrates the computation of the parameters of the axis of symmetry  $(\rho, \theta)$  in polar coordinates into the correspondence algorithm. The latter is simple and intuitive. It uses natural ordering of points of significant curvature along curves, together with constraints on the parameters  $\rho$ , and  $\theta$  to establish the correspondence. The parameters of the axis  $(\rho, \theta)$  are then evaluated from the positions of the corresponding pairs of points of significant curvature. This technique is robust and copes particularly well with noise as shown by the experimental results.*

**Keywords:** *Axis of symmetry, Correspondence, Matching, Curvature, High Curvature Points, Curves.*

## 1 Introduction

A very widely used characteristic of objects are their symmetry sets. They have been used in image description, object recognition, stereoscopic matching, viewpoint-invariant representations [12], and as shape descriptors in model based recognition [6], *et cetera*. Axis of symmetry, in particular, have been used to segment MR images of the brain into the left and right hemispheres [11]. They are also used in Robotics for picking up objects.

In this paper we present two algorithms: The first one is a fast technique for computing points of significant curvature. The second one is a robust algo-

rithm for computing axis of symmetry of an object, and which uses the points of significant curvature detected by the first.

## 2 Previous techniques

Previous techniques for computing symmetry axes can be divided into two categories: those that use segmented images and those that don't. In [1] a voting technique is applied to non-segmented images to determine axes of skewed symmetries using local skewed symmetries. This technique suffers, however, from the number of steps necessary to yield the parameters of axes. More importantly, it suffers from the use of curvature values which are known to be sensitive to noise. In [9], moments of objects in segmented images are used to compute the axis of symmetry. Relative invariants computed from bitangents in objects are used to determine symmetry in [10]. One way of extracting symmetry sets, in general, and axes of symmetry, in particular, of segmented images is presented in [4], it consists of computing the locus of centers of circles bitangent to a plane curve. Our technique uses boundaries/curves in segmented images to determine axis of symmetry of objects.

## 3 Estimating curvature

Points of maximum curvature are commonly believed to be the most perceptually significant points on digital curves, and as such have been used as shape features in both 2-D and 3-D object recognition by many researchers.

In real Euclidean plane, curvature is defined as the rate of change of slope as function of an arc length  $d\theta/ds$  where  $\theta$  is the slope and  $s$  is the arc length. For the curve  $y = f(x)$  this may be expressed as:

$$\frac{\frac{d^2y}{dx^2}}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}} \quad (1)$$

For the digital case it is not clear how to define an equivalent measure of slope as it includes computing second derivatives, and thus is very sensitive to noise.

A wide variety of point curvature estimators have been proposed such as curvature estimation from cubic fitting [3], and curvature estimation using smoothed k-cosines [7]. Evaluation of point curvature estimators was presented in [5]. These methods require a lot of computations as they try to estimate the exact value of the curvature. The method presented here gives an indication of the curvature without really computing the curvature. It yields the locations of points of significant curvature based on ratio of distances between two points  $A$  and  $B$  (arc distance  $|\widehat{AB}|$  and Euclidean distance  $|AB|$ ). It requires no computations of second derivatives and hence is very simple and efficient.

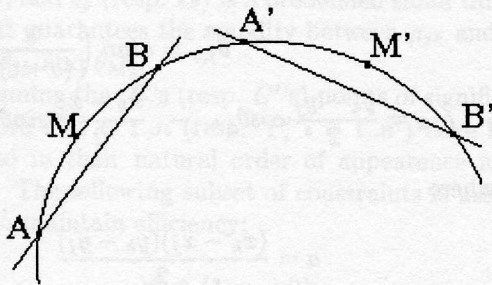


Fig. 1.  $M$  is the point under focus. The larger the ratio  $\frac{|\widehat{AB}|}{|AB|}$  the higher the curvature at point  $M$ .

Consider points  $A$  and  $B$  on a curve  $C$  (see Fig. 1). Let  $|\widehat{AB}|$  be the length of the arc<sup>1</sup>  $\widehat{AB}$  and  $|AB|$  the Euclidean distance between  $A$  and  $B$ . Let  $M$  be the mid-point of  $\widehat{AB}$  on the arc. We can have a good estimation of the curvature at point  $M$  by considering the ratio  $\frac{|\widehat{AB}|}{|AB|}$ . Thus, the larger this ratio, the higher the curvature at point  $M$ .

<sup>1</sup>This distance is computed from the 8-connected chain code of the curve  $\widehat{AB}$ .

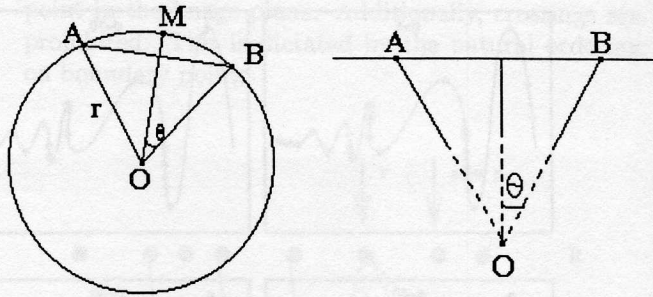


Fig. 2.  $M$  is the point under focus. The larger the ratio  $\frac{|\widehat{AB}|}{|AB|}$  the higher the curvature at point  $M$ . For the line  $(AB)$   $O$  and  $r$  go to infinity and thus  $\theta$  goes to 0

To illustrate this idea, consider a circle  $C(O, r)$  (see Fig. 2).

We have:  $|\widehat{AB}| = 2\theta r$  and  $|AB| = 2r \sin\theta$ . Hence,

$$\frac{|\widehat{AB}|}{|AB|} = \frac{\theta}{\sin\theta} \quad (2)$$

Thus, for a given  $\theta$  (i.e., for a fixed  $d = |AB|$ ), all points on a circle have the same curvature significance. Now, consider a line  $(AB)$  (see Fig. 2).  $r$  goes to infinity,  $\theta$  goes to 0 and the ratio in (2) goes to one.

$$\lim_{\theta \rightarrow 0} \frac{\theta}{\sin\theta} = 1 \quad (3)$$

Thus, given an appropriate threshold  $T > 1$  points where  $\frac{|\widehat{AB}|}{|AB|} > T$  are points of significant curvature. Formally, the algorithm is as follows:

1. For each pair of points  $(A, B)$  on the curve such that  $|AB| = d$ ,
2. If  $\frac{|\widehat{AB}|}{|AB|} > T$  declare the mid-point ( $M$ ) on the arc  $\widehat{AB}$ , point of high curvature.

$d$  and  $T$  are global parameters of the algorithm.

This method has the advantage of being very simple and easy to implement. A series of experiments are shown in Fig. 3.

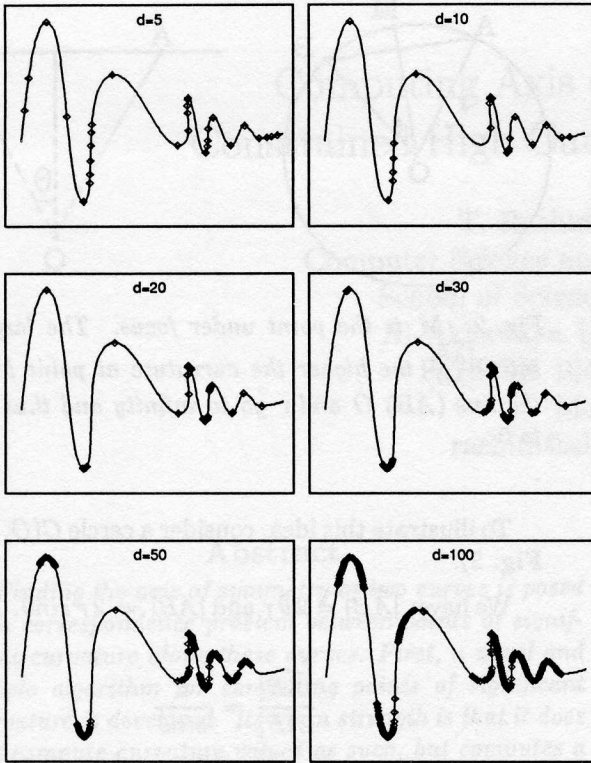


Fig. 3. Results of the algorithm on an artificial curve for various values of  $d$ . The diamonds represent the points of significant curvature detected by the algorithm.  $20 \leq d \leq 30$  is a range of optimal values.

Obviously smaller values of  $d$  yield many non maximum curvature points as is the case of  $d = 5$ . However, as  $d$  increases these erroneous points are no longer detected.  $20 \leq d \leq 30$  represent the optimal values for which almost only the maximum curvature points are detected. As  $d$  continues to increase past this range of optimal values, points in the neighbourhood of the maximum curvature points start being detected as is the case in  $d = 50$ .

At this stage, it is important to note that the algorithm for detecting the axis of symmetry of two curves is based on these detected high curvature points. This algorithm intrinsically copes with points of significant curvature other than maximum curvature points.

#### 4 Finding axis of symmetry

A symmetry axis is a line whose equation in polar form is:

$$y \cos \theta - x \sin \theta + \rho = 0 \quad (4)$$

Our goal is to find  $\rho$  and  $\theta$  parameters of the axis of symmetry for two given curves  $C$  and  $C'$  that represent object boundaries in segmented images.

Let  $(P_j, P'_j), (P_k, P'_k)$  two pairs of corresponding<sup>2</sup> points on curves  $C$  and  $C'$  (see Fig. 4).

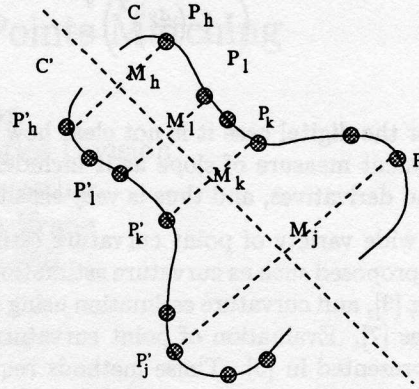


Fig. 4. Parameters  $\rho_{j,k}$  and  $\theta_{j,k}$  are computed from points  $M_j$  and  $M_k$  which are derived from corresponding points  $(P_j, P'_j)$  and  $(P_k, P'_k)$ .

Let  $M_k = (x_k, y_k)^T$  and  $M_j = (x_j, y_j)^T$  be the respective midpoints of segments  $[P_k, P'_k]$  and  $[P_j, P'_j]$ . Stating that the axis of symmetry goes through  $M_k$  and  $M_j$  gives the following parameters  $\theta_{j,k}$  and  $\rho_{j,k}$ :

$$\theta_{j,k} = \text{atan} \left( \frac{a}{(b+c)} \right) \quad (5)$$

$$\rho_{j,k} = \frac{y_k - y_j}{2} \cos \theta_{j,k} - \frac{x_k - x_j}{2} \sin \theta_{j,k} \quad (6)$$

where

$$a = \frac{(x_k - x_j)(y_k - y_j)}{2} \quad (7)$$

$$b = \frac{(x_k - x_j)^2 - (y_k - y_j)^2}{8} \quad (8)$$

$$c = \sqrt{a^2 + b^2} \quad (9)$$

A simple mathematical derivation shows that the axis of symmetry still goes through points  $M_k$  and  $M_j$  when the symmetry is skewed.

The parameters  $\rho_{j,k}$  and  $\theta_{j,k}$  must be equal for every pair of corresponding points  $(P_j, P'_j)$  and  $(P_k, P'_k)$  because there is only one axis of symmetry. This can be expressed as follows:

$$\forall j, k \quad \rho_{j,k} = Cte = \rho \quad (10)$$

$$\theta_{j,k} = Cte' = \theta \quad (11)$$

<sup>2</sup>By corresponding, we mean symmetrical.

It follows that, if correspondence between points of curves  $C$  and  $C'$  is established, the parameters  $\rho$  and  $\theta$  of the axis of symmetry can be computed using equations (5)–(9) and any pair of corresponding points. In fact, these parameters are computed using Least Median Squares regression, which has proven to be a robust technique when there is a large percentage of outliers, applied to midpoints  $(M_i)_{i=1..n}$ .

#### 4.1 Moving the goal post

With this in mind, finding the axis of symmetry is posed as establishing correspondence between points of significant curvature. In computing correspondence, equations (10) and (11) play a fundamental role. These equations simply state that amongst all the possible pairings, we should retain the pairing  $(P_1, P'_1), (P_2, P'_2) \dots$  that satisfies the following constraints:

$$\forall j, k \quad \forall h, l$$

$$|\rho_{jk} - \rho_{hl}| \leq \epsilon_\rho \quad (12)$$

$$|\theta_{jk} - \theta_{hl}| \leq \epsilon_\theta \quad (13)$$

where  $(\rho_{j,k}, \theta_{j,k})$  and  $(\rho_{h,l}, \theta_{h,l})$  are obtained from points  $(M_j, M_k)$ , and  $(M_h, M_l)$  respectively (see Fig. 4); and  $\epsilon_\rho$  (resp.  $\epsilon_\theta$ ) is a predefined small threshold that guarantees the equality between  $\rho_{jk}$  and  $\rho_{hl}$  (resp.  $\theta_{jk}$  and  $\theta_{hl}$ ).

Assuming that  $C$ 's (resp.  $C'$ 's) points of significant curvature  $P_i$   $i \in 1..n$  (resp.  $P'_i$   $i \in 1..n'$ ) have been detected in their natural order of appearance along curves. The following subset of constraints is used in order to maintain efficiency:

$$\forall i \in \{1..n - 2\},$$

$$|\rho_{i,i+1} - \rho_{i+1,i+2}| \leq \epsilon_\rho \quad (14)$$

$$|\theta_{i,i+1} - \theta_{i+1,i+2}| \leq \epsilon_\theta \quad (15)$$

where  $\rho_{i,i+1}$  and  $\theta_{i,i+1}$  are obtained from points  $M_i$  and  $M_{i+1}$  and  $\rho_{i+1,i+2}$  and  $\theta_{i+1,i+2}$  are obtained from points  $M_{i+1}$  and  $M_{i+2}$ .

The technique of integrating unknown parameters as constraints to correspondence has been termed Constrained Point Matching and has been successfully applied to computing the parameters of 2D motion of curves in successive frames [8].

Matching between points of significant curvature must be one-to-one. This constraint stems from the physical nature of these points, i.e., a point of high curvature in the scene always projects into a unique

point in the image plane. Additionally, crossings are prohibited. This is dictated by the natural ordering on boundary points.

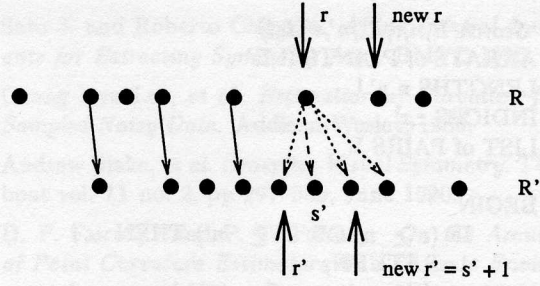


Fig. 5. Establishing correspondence between high curvature points. Point  $r$  is currently under focus. Among the  $N$  possible pairings (marked with dashed arrows),  $s'$  is the pairing that give a point  $m$  consistent with the other points  $m'$  (the middle of previously established pairs marked with solid arrows). The search continues with the point marked  $\text{new } r$  under examination, and with  $r' = s' + 1$ .

The final and desired pairing is the longest pairing between points of significant curvature of each curve, such that the parameters of the axis of symmetry obtained from any subset of it lead to the same parameters  $\rho$  and  $\theta$ .

#### 4.2 Establishing correspondence

Let  $C$  and  $C'$  be two curves,  $R$  and  $R'$  be the arrays where high curvature points of  $C$  and  $C'$  are stored. At least two high curvature points have to be present in  $C$  and  $C'$ . Curves which do not satisfy this condition are ignored. To find the axis of symmetry of  $C$  and  $C'$  we have to find the longest correct match between points from  $R$  and  $R'$  and compute mid-points of the pairs yield by the matching. To this end, an exhaustive search is employed. We proceed as follows: let  $L$  be the list of mid-points (symmetry set) at a given stage. Initially  $L$  is empty. Let  $r$  and  $r'$  be the indices of the current curvature points of  $R$  and  $R'$  under focus (see Fig. 5). For all points between  $r'$  and  $r' + N$ , select  $s'$  such that the mid-point of the new pair  $(R[r], R'[s'])$  is consistent<sup>3</sup> with all existing points in  $L$ . If such point exists, add the new mid-point to  $L$ .

If  $\#L > \#B$ , (where  $B$  is the previously best recorded set of middle points, and  $\#B$  its size), record  $L$  as the new best symmetry set, and continue with  $r = r + 1$  and  $r' = s' + 1$ . If, however, no point be-

<sup>3</sup>see below for a better understanding of consistency.

tween  $r'$  and  $r' + N$  can be constantly paired with  $r$ , continue with  $r = r + 1$ .  $N$ , the number of potential points from  $R'$  under scrunting each time, is a global parameter of the process. The algorithm is as follows:

```

Search( $R, n, r, R', n', r', L, l$ )
ARRAYS of POINTS  $R, R'$ 
LENGTHS  $n, n', l$ 
INDICES  $r, r'$ 
LIST of PAIRS  $L$ 

BEGIN
  IF ( $r \geq n$  OR  $r' \geq n'$ ) THEN
    RETURN;

  FOR  $i=r'$  TO  $r'+N$  DO
    IF Consistent( $R[r], R'[i], L, l$ ) THEN
       $L \leftarrow L \cup \{\text{mid}(R[r], R'[i])\}$ ;
      Search( $R, n, r+1, R', n', i+1, L, l+1$ );
      IF Improvement( $L, B$ ) THEN
         $B = L$ ;
      END
    END
  END
END
END

```

Algorithm 4.1. *Constrained Point Matching Algorithm*

*Improvement* ( $L, B$ ) determines whether the symmetry set in  $L$  is better than the one in  $B$ . Clearly, if  $\#L > \#B$  there is improvement, otherwise there is none.

The function *Consistent*( $x, y, L, l$ ) returns 1 if the mid-point  $P = (x, y)^T$  is consistent with the list of previously established mid-points in  $L$  (of length  $l$ ) at the time of the call. Let  $P_{i-1}$  and  $P_i$  the two last points in  $L$ . To check the consistency, we compute the parameters  $\rho_{i-1, i}$  and  $\theta_{i-1, i}$  of the line  $(P_{i-1} P_i)$  and  $\rho_e$  and  $\theta_e$  of the line  $(P_i P)$ . If  $|\rho_{i-1, i} - \rho_e| < \epsilon_\rho$  and  $|\theta_{i-1, i} - \theta_e| < \epsilon_\theta$  then  $P$  is consistent with the points in  $L$  and *Consistent*() returns 1 otherwise 0.

The final parameters  $\rho$  and  $\theta$  of the symmetry axis are computed using Least median squares regression applied to the longest consistent pairs of points  $(B[i])_{i=1..n}$ .

The number of pairs obtained is a good indication of the correctness of the computed parameters of the line. The ratio  $r = \frac{\#B}{\min(\#R, \#R')}$  is a self verifying indicator of the goodness of the line model  $(\rho, \theta)$ . The computed parameters of the axis of symmetry are taken to be reliable if  $r \geq 0.5$ , which implies that 50% of the points of significant curvature are correctly matched in accordance with the constraints.

## 5 Experiments and results

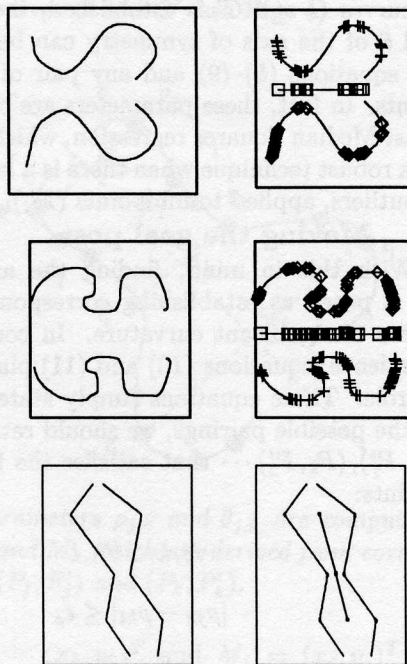


Fig. 6. Pairs of symmetric artificial curves (left). The points corresponding to the best match and the axis of symmetry are sketched (right). Note that the fifth and sixth figures are of a skewed artificial curve.

Fig. 6 shows results obtained for artificial curves. For the three experiments the parameters that have been used are  $N = 10$ ,  $\epsilon_\rho = 1$  pixel, and  $\epsilon_\theta = 0.1$  rad. The figure shows the high curvature points that have been matched along the curves together with the computed axes. The third experiment shows the result of the algorithm on skewed curves. The self verifying factor  $r$  for the first, second and third experiment are respectively 0.58, 0.92, and 0.86.

Fig. 7 shows the results obtained for a real image which has been segmented using a succession of Space's edge detector [14] and Rachidi's construction of coherent boundaries [13]. The parameters obtained for this image are summarised in the following table.

Object	$\rho$ (pixels)	$\theta$ (rad)
1	-20.07	1.51
2	-19.4	1.15
3	-40.5	1.15
4	-89	1.56

Table 1. Objects are numbered from top to bottom. The parameters used to compute  $\rho$  and  $\theta$  are:  $d = 20$ ,  $N=10$ ,  $\epsilon_\theta = 0.1$  rad and  $\epsilon_\rho = 1$  pixel.

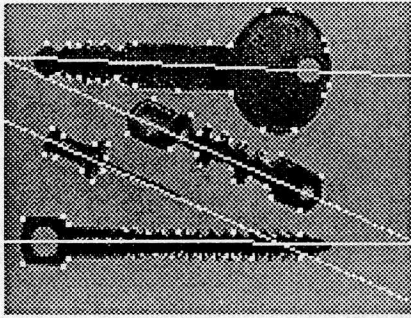


Fig. 7. The axes obtained for each object together with the high curvature points used in their computation are superimposed on the original image (see Table 1 for the parameters of the axis of symmetry).

Finally, Fig. 8 shows the axis found for the skewed dominant object. The same parameters i.e.,  $d = 20$ ,  $N = 10$ ,  $\epsilon_\theta = 0.1$  rad and  $\epsilon_\rho = 1$  pixel have been used. The parameters of the axis of symmetry are  $\theta = 1.55$  rad and  $\rho = -74.8$  rad.

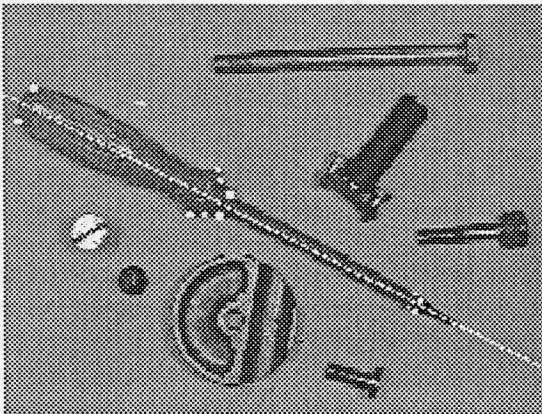


Fig. 8. The axis obtained for the dominant skewed object together with the high curvature points used in its computation are superimposed on the original image.

## Conclusion

The various experiments demonstrate the robustness of the approach presented in this paper to determine the parameters of symmetry axis of two curves. This robustness stems from the formulation of the parameters as high curvature points pairings rather than curvature-dependent expressions, and from the ability of the algorithm to overcome the non-maximum curvature points that may be detected at the earlier stage. The authors are currently investigating the usage of this technique in computing other symmetry sets than

axes.

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