

# $H(\omega) \Rightarrow$ MRA: Theoretical Analysis and Application to Pattern Recognition and Document Analysis

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## Abstract

From a multiresolution analysis (MRA), the scaling function  $\varphi(x)$  can be derived, the transfer function  $H(\omega)$  and frequency response  $\{h_k\}$  can be obtained further. Thus, a bank of filters will then be designed. This scheme can be symbolized by  $MRA \Rightarrow H(\omega)$ . Unfortunately, the MRA and  $\varphi(x)$  can't be given beforehand in most of cases, and many scaling function  $\varphi(x)$  and wavelet function  $\psi(x)$  can't be written by analytic expressions. Therefore, as an alternative, first, we can find  $H(\omega)$  and  $\{h_k\}$  according to the requirement given in a specific application.  $\varphi(x)$  will then be deduced, and the MRA will further be obtained. This scheme can be symbolically denoted by  $H(\omega) \Rightarrow MRA$ . This paper presents a complete theoretical analysis of it. As the applications of the scheme of  $H(\omega) \Rightarrow MRA$  to the field of pattern recognition including document processing, several practical examples are also presented in this paper.

## 1 Introduction

With respect to the conventional multiresolution analysis (MRA) theory, there exists a sequence of closed subspaces  $\{V_j\}_{j \in \mathbb{Z}}$  of  $L^2(\mathbb{R})$ , and furthermore, there exists a function  $\varphi(x) \in V_0$  such that  $\{\varphi(x - k), k \in \mathbb{Z}\}$  becomes the orthonormal bases of  $V_0$ . Here, the function  $\varphi(x)$  is termed *scaling function*. According to such a MRA theory, one can derive the *transfer function*  $H(\omega)$  and its corresponding *frequency response function*  $\{h_k\}$ , from the two-scale relation. And, as one step further, one

can also define the *wavelet function*  $\psi(x)$  and  $\{g_k\}$  that have in turn been utilized in the commonly-used Mallat algorithm [8].

Let  $\{c_{j,k}, k \in \mathbb{Z}\}$  be the input discrete digital signal, which can be decomposed into a set of sub-signals by the following equations:

$$c_{j-1,m} = \sum_k \bar{h}_k c_{j,k+2m}, \quad d_{j-1,m} = \sum_k \bar{g}_k c_{j,k+2m}$$

where  $\{c_{j-1,m}, m \in \mathbb{Z}\}$  denotes the output discrete low-frequency digital signal, while  $\{d_{j-1,m}, m \in \mathbb{Z}\}$  denotes the output discrete high-frequency digital signal. What the above equations actually imply is that once MRA is given, it is possible to construct  $H(\omega)$  and  $\{h_k\}$ . Moreover, from the obtained  $H(\omega)$  or  $\{h_k\}$ , Mallat algorithm can readily be applied. This conventional MRA theory may be symbolically expressed as follows:

$$MRA \Rightarrow H(\omega).$$

We should note that here  $H(\omega)$  is one of the most important notions that has played a key role in the entire formalism of the MRA theory. In other words, it is often assumed that MRA is given,  $H(\omega)$  is desired. In fact, two of the fundamental functions involved in MRA are: scaling function  $\varphi(x)$  and wavelet function  $\psi(x)$ . The scaling function  $\varphi(x)$  operates on the low-frequency components of a signal, as a low-pass filter in practical applications, while the wavelet function  $\psi(x)$  performs as high-pass filter to the high-frequency components of the signal.

However, in practice, the MRA and  $\varphi(x)$  can't be given beforehand in most of cases. Mean-

while, many scaling function  $\varphi(x)$  and wavelet function  $\psi(x)$  can't be written by analytic expressions. These obstacles obstruct the applications of the above scheme.

Let us consider the relationships of  $H(\omega)$  with  $\varphi(x)$  and  $\psi(x)$ , which can be described, respectively, as follows:

$$\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} H(2^{-j}\omega), \quad \hat{\psi}(\omega) = G(\omega/2) \prod_{j=2}^{\infty} H(2^{-j}\omega)$$

The above expressions have in fact implied that scaling function  $\varphi(x)$  and wavelet function  $\psi(x)$  can both be written in terms of transform function  $H(\omega)$ .

Naturally, this has led us to wonder the inverse problem of deriving  $H(\omega)$  from MRA. In the inverse scheme, we first find  $H(\omega)$  by the requirement given in a specific application, and then define  $\varphi(x)$  as  $\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} H(2^{-j}\omega)$ . Furthermore, we define  $V_0 = \text{span}\{\varphi(x-k), k \in \mathbb{Z}\}$ ,  $V_j = \{f, f(2^{-j}x) \in V_0\}$ , and prove that the derived  $\{V_j\}_{j \in \mathbb{Z}}$  is exactly a multiresolution analysis of  $L^2(\mathbb{R})$ . In what follows, we shall examine in details the scheme of  $H(\omega) \Rightarrow \text{MRA}$  which is the core of this paper.

In this paper, we will proceed with our discussions in addressing the following three problems, namely: (1).  $H(\omega) \Rightarrow \varphi(x)$ , (2).  $(\cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2})^{2N-1} \Rightarrow H(\omega)$ , (3).  $\varphi(x) \Rightarrow V_j$ . These will be described in Sections 2 - 4. As the applications of the scheme of  $H(\omega) \Rightarrow \text{MRA}$  to the field of pattern recognition including document processing, the following practical examples will be presented in the remainder of this paper:

- Application 1 : Two-dimensional wavelet with  $N = 1$ ,  $\cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2} = 1$  extracting multiresolution features in recognition of handwritten numerals;
- Application 2 : Two-dimensional wavelet with  $N = 5$ ,  $(\cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2})^9 = 1$  extracting reference lines in form documents processing;

They will be presented in Sections 5-6 respectively.

## 2 $H(\omega) \Rightarrow \varphi(x)$

It can be observed from the preceding section that if we assume that MRA is given (that is also to say,  $\varphi(x)$  is given first), then the Fourier transform of the scaling function  $\varphi(x)$  can be written as follows:

$$\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} H(2^{-j}\omega).$$

This also implies that given MRA, the filter function family  $\{H(2^{-j}\omega), j \geq 1\}$  as derived from a single filter function  $H(\omega)$  by a binary dilation will have a convergent product  $\prod_{j=1}^{\infty} H(2^{-j}\omega)$ . Thus,  $\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} H(2^{-j}\omega)$  is well defined. Note that here  $H(\omega)$  is derived from  $\varphi(x)$ . Now, let us pose the inverse problems:

- Suppose that  $H(\omega)$  is given, under what condition of  $H(\omega)$ , we can guarantee that  $\prod_{j=1}^{\infty} H(2^{-j}\omega)$  converges, and consequently,  $\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} H(2^{-j}\omega)$  is well defined?
- Moreover, under what condition, we can guarantee that  $\hat{\varphi}(\omega)$  is an square-integral continuous function, i.e.,  $\hat{\varphi}(\omega) \in L^2(\mathbb{R}) \cap C(\mathbb{R})$ ?

where  $C(\mathbb{R})$  denotes all the continuous functions over  $\mathbb{R}$ . Note that conditions

$$\hat{\varphi}(\omega) \in L^2(\mathbb{R}), \quad \text{and} \quad \hat{\varphi}(\omega) \in C(\mathbb{R}),$$

are of utmost importance. The reasons are below:

- The condition of  $\hat{\varphi}(\omega) \in L^2(\mathbb{R})$  can guarantee that the scaling function  $\varphi(x)$  belongs to the space  $L^2(\mathbb{R})$ , namely

$$\varphi(x) \in L^2(\mathbb{R}).$$

Consequently, we can consider that how the orthonormal bases in space  $L^2(\mathbb{R})$  can be constructed from  $\{\varphi(x-k), k \in \mathbb{Z}\}$ .

- Combination of conditions of

$$\hat{\varphi}(\omega) \in C(\mathbb{R}), \quad \text{and} \quad \hat{\varphi}(0) > 0,$$

can guarantee that there exists a neighbourhood  $(-c, c)$  which contains the point of zero, so that

$$\hat{\varphi}(\omega) \neq 0, \quad \text{for} \quad \omega \in (-c, c).$$

This characteristic will be used in the coming proofs.

The remainder of this sub-section will be served to solve the above problems. First, we discuss the convergence of  $\prod_{j=1}^{\infty} H(2^{-j}\omega)$ , namely, under what condition of  $H(\omega)$ , we can guarantee that  $\prod_{j=1}^{\infty} H(2^{-j}\omega)$  converges? We have the following theorem [2]:

**Theorem 2.1** Let  $H(\omega)$  satisfy the following two conditions:

$$H(\omega) \in C(\mathbb{R}), \quad (1)$$

$$H(\omega) = 1 + O(|\omega|^\epsilon), \quad \text{for some } \epsilon > 0, \quad (2)$$

then,  $\prod_{j=1}^{\infty} H(2^{-j}\omega)$  is uniform converged in any compact set, furthermore, a continuous function  $\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} H(2^{-j}\omega)$  can pointwise be defined in  $\mathbb{R}$ .

Before the proof of Theorem 2.1, two basic terms will be explained first, namely:

- The notation  $O(|\omega|^\epsilon)$  denotes an infinitesimal of same order of  $|\omega|^\epsilon$ , i.e.  $\lim_{|\omega| \rightarrow 0} \frac{O(|\omega|^\epsilon)}{|\omega|^\epsilon} = c$ , where,  $c$  is a constant.
- The term of *compact set* is a very important concept in the mathematical subject of *Topology*. Its mathematical definition is strict which is not necessary to give to this paper for readers of scientists and engineers. What we should point out here is that in the case of a special topological space, the one-dimensional real space, a compact set can be considered as a bounded closed set.

In Theorem 2.1, we require  $H(\omega)$  is a continuous function only. However, there exist a number of types of continuous functions, and this theorem neither point out what kind of  $H(\omega)$  will be needed nor what form of continuous function will be  $H(\omega)$  look like. Meanwhile, Eq. (2) is only a descriptive condition, which will be difficult to verify for the scientists and engineers who don't have strong mathematic background. Thus, the conditions in Theorem 2.1 will be changed as follows in order to make the conditions stronger:

- A special type of periodical continuous function, trigonometric polynomial, with the period of  $2\pi$ , will be chosen as  $H(\omega)$
- The descriptive condition Eq. (2) will be replaced by a constructive one.

Consequently, the following Corollary 2.1 can be deduced:

**Corollary 2.1** Suppose  $H(\omega)$  satisfy:

$$H(\omega) = \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{2}} h_k e^{-i\omega k} \in C([0, 2\pi]), \quad (3)$$

$$H(0) = 1, \quad (4)$$

$$\sum_{k \in \mathbb{Z}} |h_k| |k|^\epsilon < \infty, \quad 0 < \epsilon \leq 1, \quad (5)$$

Then following holds:

$$H(\omega) = 1 + O(|\omega|^\epsilon)$$

Obviously, when  $\{h_k, k \in \mathbb{Z}\}$  is a finite-length sequence of real numbers, the conditions Eqs. (3), (4) and (5) can be verified easily.

Apart from the conditions (3) - (5), if an additional condition

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 1$$

will be joined to  $H(\omega)$ , then  $\hat{\varphi}(\omega)$  will not only be continuous but also square integrable, i.e. it follows that

$$\hat{\varphi}(\omega) \in L^2(\mathbb{R}) \cap C(\mathbb{R}).$$

Based on the above additional condition, we will add another condition

$$\Phi(\omega) = \sum_k |\hat{\varphi}(\omega + 2k\pi)|^2 \geq C > 0, \quad \text{a.e. } \omega \in [0, 2\pi]$$

to  $\hat{\varphi}(\omega)$ , such that  $\{\varphi(x - k), k \in \mathbb{Z}\}$  will be an orthonormal system. Please note that the notation "a. e." denotes "almost everywhere", it indicates that the above inequality is held for almost all  $\omega$  in the closed interval of  $[0, 2\pi]$ .

By using the above conditions, we can establish the following main result, Theorem 2.2:

**Theorem 2.2** Suppose  $H(\omega)$  satisfy:

$$H(\omega) = \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{2}} h_k e^{-i\omega k} \in C([0, 2\pi]), \quad H(0) = 1,$$

$$\sum_{k \in \mathbb{Z}} |h_k| |k|^\epsilon < \infty, \quad 0 < \epsilon \leq 1,$$

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 1 \quad (6)$$

Then, we have

$$\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} H(2^{-j}\omega) \in L^2(\mathbb{R}),$$

According to Theorem 2.2, it can be deduced that under the four conditions provided in (3), (4), (5), and (6), the necessary and sufficient condition for

$\{\varphi(x - k), k \in \mathbb{Z}\}$  constructs orthonormal system is the following requirement

$$\Phi(\omega) \geq C > 0, \quad \text{a.e. } \omega \in [0, 2\pi].$$

This will also be difficult to verify for the scientists and engineers who want apply wavelet theory to

their own fields and don't have strong mathematic background. Therefore, finding some replacement conditions, which will be verified easily, is very significant for the applications of wavelet theory. Fortunately, reference [6] guarantees that *Cohen Condition* can be used for this purpose, namely, the condition (??) can be replaced by Cohen condition. The definition of Cohen condition will be presented below:

**Definition 2.1** A set  $K \subset \mathbb{R}$  is said to be  $2\pi$ -congruent to  $T$ , if  $|K| = 2\pi$ , and there exists an  $\alpha \in \mathbb{Z}$ , such that  $\omega + 2\pi\alpha \in K$ , for any  $\omega \in T$ .  $H(\omega) \in C(T)$  is said to satisfy the Cohen condition, if there exists a compact set  $K$  which is congruent to  $T$ , and 0 is an inner point of  $K$ , such that

$$\inf \{ |H(2^{-j}\omega)| : j \geq 1, \omega \in K \} \geq C_0 > 0$$

By applying the above definition, we have the following result:

**Theorem 2.3** When  $H(\omega)$  satisfy:

$$H(\omega) = \sum_{k \in \mathbb{Z}} \frac{1}{\sqrt{2}} h_k e^{-i\omega k} \in C([0, 2\pi])$$

$$\sum_{k \in \mathbb{Z}} |h_k| |k|^\epsilon < \infty, \quad 0 < \epsilon \leq 1$$

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = 1.$$

Then, we have the following from Theorem 2.2

$$\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} H(2^{-j}\omega) \in L^2(\mathbb{R}) \cap C(\mathbb{R}).$$

In addition, the following relations are true:

- (i) Let  $H(\omega)$  satisfy the Cohen condition, then  $\{\varphi(x - k), k \in \mathbb{Z}\}$  constructs an orthonormal system.
- (ii) Suppose  $\{\varphi(x - k), k \in \mathbb{Z}\}$  constructs an orthonormal system, and  $\Phi(\omega) = \sum_k |\hat{\varphi}(\omega + 2k\pi)|^2$  is continuous, then  $H(\omega)$  satisfies the Cohen condition.

The question is that whether such a function  $H(\omega)$  exists so that the above five condition listed in the box are satisfied, and if so, how do we find it? I. Daubechies answered this question using the Riesz Lemma in [1]. In this paper, such  $H(\omega)$  will be constructed directly from trigonometric identity

$$\left( \cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2} \right)^{2N-1} = 1 \quad (7)$$

instead of repeating Daubechies' work. In the following sub-section, we will give the details of  $(\cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2})^{2N-1} \Rightarrow H(\omega)$ , and verify that this  $H(\omega)$  will satisfy the conditions in Theorem 2.3.

$$3 \quad \left( \cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2} \right)^{2N-1} \Rightarrow H(\omega)$$

In this sub-section, we will examine the trigonometric identity

$$\left( \cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2} \right)^{2N-1}, \quad N = 1, 2, 3, \dots,$$

according to a stage that studies from the concrete examples to abstract concepts, and from special cases to general ideas. We begin with a concrete example to construct a function  $H(\omega)$ , where we take  $N = 2$  in the trigonometric identity. Thus, we have the following construction:

**Construction 1**

It is clear that the value of the trigonometric identity is one, namely,

$$1 = \left( \cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2} \right)^3.$$

If let

$$|H(\omega)|^2 = \cos^6 \frac{\omega}{2} + 3 \cos^4 \frac{\omega}{2} \sin^2 \frac{\omega}{2}, \quad (8)$$

Then, we have

$$|H(\omega + \pi)|^2 = \sin^6 \frac{\omega}{2} + 3 \sin^4 \frac{\omega}{2} \cos^2 \frac{\omega}{2}.$$

However, there exist many  $H(\omega)$ s, such that Eq. (8) is satisfied. A similar example is that  $a \neq -a$ , but  $a^2 = (-a)^2$ . The problem we should discuss is that, how an appropriate function  $H(\omega)$  can be chosen.

It is obvious that the condition  $|H(\omega)|^2 + |H(\omega + \pi)|^2 = 1$  is held for all cases. Next, because of the following formulae are true:

$$\frac{1 + e^{-i\omega}}{2} = \frac{e^{-i\omega/2}(e^{i\omega/2} + e^{-i\omega/2})}{2} = e^{-i\omega/2} \cos \frac{\omega}{2}, \quad (9)$$

$$\frac{1 - e^{-i\omega}}{2} = \frac{e^{-i\omega/2}(e^{i\omega/2} - e^{-i\omega/2})}{2} = ie^{-i\omega/2} \sin \frac{\omega}{2}. \quad (10)$$

Then, the function  $H(\omega)$  can be chosen as

$$H(\omega) = \sum_{k=0}^3 \frac{1}{\sqrt{2}} h_k e^{-i\omega k}. \quad (11)$$

**Theorem 3.1** Let  $H(\omega)$  be form of

$$H(\omega) = \left( e^{-i\omega/2} \cos \frac{\omega}{2} \right)^2 e^{-i\omega/2} \left( \cos \frac{\omega}{2} + i\sqrt{3} \sin \frac{\omega}{2} \right)$$

Consequently, equation

$$\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} H(2^{-j}\omega) \in L^2(\mathbb{R}) \cap C([0, 2\pi])$$

is held, and  $\{\varphi(x - k), k \in \mathbb{Z}\}$  constructs an orthonormal system.

Similarly, we take  $N = 3$  in the trigonometric identity,  $(\cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2})^{2N-1}$ , then another  $H(\omega)$  or  $\{h_k\}$  can be obtained. The detailed process will be presented in Construction 2 as follows:

### Construction 2

Let  $N$  be 3, and the value of the trigonometric identity be one, that is

$$\begin{aligned} 1 &= \left(\cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2}\right)^5 \\ &= \cos^{10} \frac{\omega}{2} + 5 \cos^8 \frac{\omega}{2} \sin^2 \frac{\omega}{2} + 10 \cos^6 \frac{\omega}{2} \sin^4 \frac{\omega}{2} \\ &\quad + 10 \cos^4 \frac{\omega}{2} \sin^6 \frac{\omega}{2} + 5 \cos^2 \frac{\omega}{2} \sin^8 \frac{\omega}{2} + \sin^{10} \frac{\omega}{2}. \end{aligned}$$

Let us write

$$|H(\omega)|^2 = \cos^{10} \frac{\omega}{2} + 5 \cos^8 \frac{\omega}{2} \sin^2 \frac{\omega}{2} + 10 \cos^6 \frac{\omega}{2} \sin^4 \frac{\omega}{2}, \quad (12)$$

then, we have

$$|H(\omega + \pi)|^2 = \sin^{10} \frac{\omega}{2} + 5 \sin^8 \frac{\omega}{2} \cos^2 \frac{\omega}{2} + 10 \sin^6 \frac{\omega}{2} \cos^4 \frac{\omega}{2}.$$

Recalling Eqs. (9) and (10) yields

$$H(\omega) = \sum_{k=0}^5 \frac{1}{\sqrt{2}} h_k e^{-i\omega k}.$$

Therefore,  $\{h_k\}$  can be constructed as follows:

$$\begin{aligned} h_0 &\approx 0.332670552 & h_1 &\approx 0.806891509 \\ h_2 &\approx 0.459877502 & h_3 &\approx -0.135011020 \\ h_4 &\approx -0.085441273 & h_5 &\approx -0.035226291 \end{aligned} \quad (13)$$

It is similar to Theorem 3.1, the requirements

$$H(\omega) \in C([0, 2\pi]),$$

$$|H(\omega)|^2 + |H(\omega + \pi)|^2 = \left(\cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2}\right)^5 = 1,$$

$$H(0) = 1,$$

$$\sum_{k \in \mathbb{Z}} |h_k| |k|^\epsilon \leq \sum_{k=0}^5 |h_k| \cdot k < \infty, \quad 0 < \epsilon \leq 1,$$

are satisfied for  $H(\omega) = \sum_{k=0}^5 \frac{1}{\sqrt{2}} h_k e^{-i\omega k}$  which has been constructed in Construction 2.

Additionally, from Eq. (12), it follows that

$$|H(\omega)| \geq \left| \cos^3 \frac{\omega}{2} \right|.$$

When  $\omega \in K = [-\pi, \pi]$ , since  $\omega/2^{j+1} \in [-\pi/2^{j+1}, \pi/2^{j+1}]$ , it is clear that

$$|H(2^{-j}\omega)| \geq |\cos^3(\omega/2^{j+1})| \geq \left| \cos^3 \frac{\pi}{4} \right| = \frac{\sqrt{2}}{4}, \quad j \geq 1.$$

This indicates that

$$\inf \{|H(2^{-j}\omega)| : j \geq 1, \omega \in [-\pi, \pi]\} \geq \frac{\sqrt{2}}{4} > 0.$$

Therefore, the Cohen Condition is satisfied for  $H(\omega)$  in Construction 2. By applying Theorem 2.3, we can conclude that  $\hat{\varphi}(\omega)$  which has been defined from  $H(\omega)$  in Construction 2 is continuous over space of  $L^2(\mathbb{R})$ , namely,  $\hat{\varphi}(\omega) \in L^2(\mathbb{R}) \cap C([0, 2\pi])$ , and  $\{\varphi(x - k), k \in \mathbb{Z}\}$  constructs an orthonormal system.

From the above analysis, it can be found that the functions,  $H(\omega)$ s, which are constructed by using trigonometric identity,  $(\cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2})^{2N-1}$ , can be classified into two categories:

- $N = 1$ ,  $H(\omega) = \frac{1+e^{-i\omega}}{2} = \sum_{k=0}^1 \frac{1}{\sqrt{2}} h_k e^{-i\omega k}$ , this is the filtering function for Harr wavelet.
- $N \geq 2$ ,  $H(\omega) = \sum_{k=0}^{2N-1} \frac{1}{\sqrt{2}} h_k e^{-i\omega k}$ , this is the filtering function for the well-known Daubechies wavelet.

## 4 $\varphi(x) \Rightarrow V_j$

We have seen that in the previous sections, we start with an elementary mathematic formula  $1 = (\cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2})^{2N-1}$ , then, the trigonometric poly-

nomial  $H(\omega) = \sum_{k=0}^{2N-1} \frac{1}{\sqrt{2}} h_k e^{-i\omega k}$  is constructed,

from which the function  $\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} H(2^{-j}\omega)$  is well

defined, such that  $\hat{\varphi}(\omega) \in L^2(\mathbb{R}) \cap C([0, 2\pi])$ , and finally,  $\{\varphi(x - k), k \in \mathbb{Z}\}$  constructs an orthonormal system.

Note that, "orthonormal system" and "orthonormal basis" are not equivalent, the difference between them is that we must consider a "space" for the later. However, we don't have it, so far. In this sub-section, we will introduce a closed subspace of  $L^2(\mathbb{R})$ , from the function  $\varphi(\omega)$ , to serve as the space for the orthonormal basis. This closed subspace is symbolize by  $V_0$ . It will be shown that  $\{\varphi(x - k), k \in \mathbb{Z}\}$  constructs orthonormal bases of space  $V_0$ . Consequently, a sequence of closed subspaces  $\{V_j\}_{j \in \mathbb{Z}}$  of  $L^2(\mathbb{R})$  can be obtained. We will further verify that  $\{V_j\}_{j \in \mathbb{Z}}$  is a multiresolution analysis (MRA) of  $L^2(\mathbb{R})$ .

First, we will provide a general discussion to indicate that under what conditions, the following

$$\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}),$$

holds for a given sequence of closed subspaces  $\{V_j\}_{j \in \mathbb{Z}}$  of  $L^2(\mathbb{R})$ ? For this purpose, we have Theorems 4.1 and 4.2 [3].

**Theorem 4.1** Let  $\{V_j\}_{j \in \mathbb{Z}}$  be increment family of closed subspaces of  $L^2(\mathbb{R})$ ,  $V_j$  be  $2^{-j}\mathbb{Z}$ -translation invariant. Then

$$\left( \bigcup_{j \in \mathbb{Z}} V_j \right)^\wedge = L^2(\Omega_0), \quad \Omega_0 = \bigcup_j \bigcup_{f \in V_j} \text{supp } f(\omega)$$

For any  $h(x) \in L^2(\mathbb{R})$ , the closed subspace  $V_j$  is defined as follows:

$$V_j = \text{span} \{h(2^j x - k), k \in \mathbb{Z}\}.$$

$V_j$  is called a *spanning space* of the function family  $\{h(2^j x - k), k \in \mathbb{Z}\}$  under  $L^2(\mathbb{R})$ , namely, it is *completeness* of the set which is a finite linear combination of the functions in the family  $\{h(2^j x - k), k \in \mathbb{Z}\}$ .

The following theorem is a very important and useful tool to solve the basic and significant question: "under what conditions,  $\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R})$  will be held?"

**Theorem 4.2** Suppose the following conditions are satisfied for function  $h(x)$ : (1),  $h(x) \in L^2(\mathbb{R})$ , (2),  $\hat{h}(\omega) \neq 0$ , a.e.  $\omega \in (-c, c)$  for some  $c > 0$ , (3),  $V_j = \text{span} \{h(2^j x - k), k \in \mathbb{Z}\}$ , (4),  $\{V_j\}_{j \in \mathbb{Z}}$  is a increment family. Then we have

$$\bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}).$$

Now, we turn to the discrimination of  $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$  by setting  $h(x) \in L^2(\mathbb{R})$ ,  $V_0 = \text{span} \{h(x - k), k \in \mathbb{Z}\}$ , and  $V_j = \{f(x); f(2^j x) \in V_0\}$ . This can be found in the following.

**Theorem 4.3** Suppose the following conditions are satisfied for function  $h(x)$ :

- (i)  $h(x) \in L^2(\mathbb{R})$ ,
- (ii)  $V_j = \text{span} \{h(x - k), k \in \mathbb{Z}\}$ ,
- (iii)  $V_j = \{f(x); f(2^j x) \in V_0\}$ .

Then we have

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\}.$$

We end this sub-section with a specific discussion to indicate that under what conditions, the following

$$\overline{\bigcup_{j \in \mathbb{Z}} V_j} = L^2(\mathbb{R}),$$

holds for a given sequence of closed subspaces  $\{V_j\}_{j \in \mathbb{Z}}$  of  $L^2(\mathbb{R})$ ?

In the Theorems 4.2 and 4.3, the function  $h(x)$  was chosen as a general square integrable function. Now, we consider a special form instead of the general one. Applying  $\varphi(x)$  to replacement of the general square integrable function establishes the following result.

**Theorem 4.4** Let the trigonometric polynomial

$$H(\omega) = \sum_{k=0}^{2N-1} \frac{1}{\sqrt{2}} h_k e^{-i\omega k}$$

be the result of solving  $|H(\omega)|^2 + |H(\omega + \pi)|^2 = (\cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2})^{2N-1}$ ,  $N = 1, 2, 3, \dots$ . By setting (1),  $\hat{\varphi}(\omega) = \prod_{j=1}^{\infty} H(2^{-j}\omega)$ ,

(2),  $V_j = \text{span} \{h(x - k), k \in \mathbb{Z}\}$ , (3),  $V_j = \{f(x); f(2^j x) \in V_0\}$ , then  $\{V_j\}_{j \in \mathbb{Z}}$  is a multiresolution analysis (MRA) of  $L^2(\mathbb{R})$ .

## 5 Application 1 : 2-D Wavelet with $\cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2} = 1$ Extracting Multiresolution Features in Recognition of Handwritten Numerals

By considering a special case,  $N = 1$  in Eq. (7), we can construct a function for a well-known wavelet called *Harr wavelet* which is adequate for local detection of line segments and global detection of line structures with fast computation. This method enables us to have an invariant interpretation of the character image at different resolutions and presents a multiresolution analysis in the form of coefficient matrices [5, 7]. Since the details of character image at different resolutions generally characterize different physical structures of the character, and the coefficients obtained from wavelet transform are very useful in recognizing unconstrained handwritten numerals. Therefore, in this application, wavelet transform with a set of Haar wavelets has been used for multiresolution feature extraction in handwriting recognition.

In this way, we take

$$1 = \cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2}.$$

The scaling function  $\varphi(x)$  and wavelet function  $\psi(x)$  can be represented by

$$\varphi(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (14)$$

and

$$\psi(x) = c_1\varphi(2x) - c_0\varphi(2x-1) = \begin{cases} 1 & \text{if } 0 < x < \frac{1}{2} \\ -1 & \text{if } \frac{1}{2} \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (15)$$

An image of the handwritten character can be decomposed into its wavelet coefficients by using Mallat's pyramid algorithm. By using Haar wavelets, an image  $F$  is decomposed. This decomposition can be archived by convolving the  $2 \times 2$  image array with the Haar masks as follows:

$$D_k^j = \frac{1}{4}I^{j+1} \otimes H_k \quad k = 1, 2, 3, 4 \quad (16)$$

where,  $\otimes$  is the convolution operator,  $D_k^j$  are decomposed images at resolution  $2^j$ ,  $I^{j+1}$  is  $2 \times 2$  image at resolution  $2^{j+1}$ , and  $H_k$  are Haar masks defined in Fig. 1. In this application, the decomposed

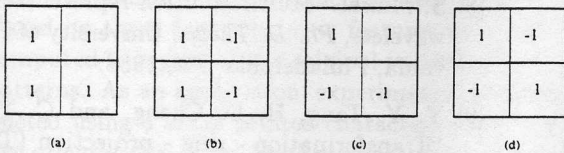


Figure 1: Haar masks

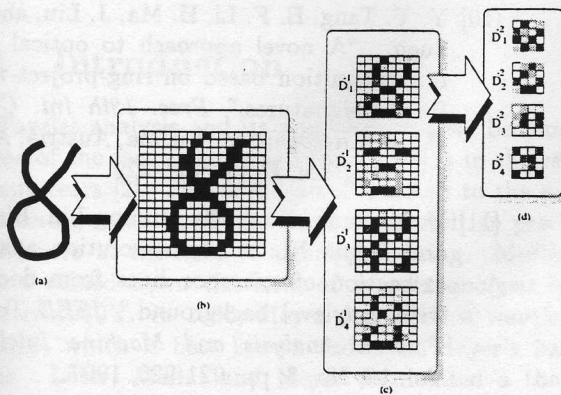


Figure 2: Overview of multiresolution feature extraction

results at resolution  $2^{-1}$  and  $2^{-2}$  are used as multiresolution features. Fig. 2 shows the process of multiresolution feature extraction. Fig. 2(a) gives an input image of the handwritten number "8", its

digitized image is shown in Fig. 2(b). The decomposed feature vector at resolution  $2^{-1}$  is illustrated in Fig. 2(c), while the decomposed feature vector at resolution  $2^{-2}$  is in Fig. 2(d).

These features have been used to recognize the unconstrained handwritten numerals from the database of Concordia University of Canada, Electro-Technical Laboratory of Japan, and Electronics and Telecommunications Research Institute of Korea. The error rates were 3.20%, 0.83%, and 0.75%, respectively. These results have been shown that the proposed scheme is very robust in terms of various writing styles and sizes. The detailed description of this application can be referred in [5].

## 6 Application 2 : 2-D Wavelet with $(\cos^2 \frac{\omega}{2} + \sin^2 \frac{\omega}{2})^9 = 1$ Extracting Reference Lines in Form Documents Processing

This example takes  $N = 5$  in Eq. (7). In general, a form document consists of straight lines which are oriented mostly in horizontal and vertical directions. These lines are called *reference lines*. The information that should be entered to computer and processed usually appears either above, beneath, or beside these reference lines. A typical example is a bank cheque, where reference lines are printed to guide the users to write in the name of the payee, amount, and date in the appropriate places. Consequently, extraction of such reference lines plays a very important role in the form processing. Wavelet decomposition algorithm, and compactly supported orthonormal wavelets can be used to transform a document image into sub-images. Based on these sub-images, the reference lines of a grey-level document can be extracted, and knowledge about the geometric structure of the document can be acquired.

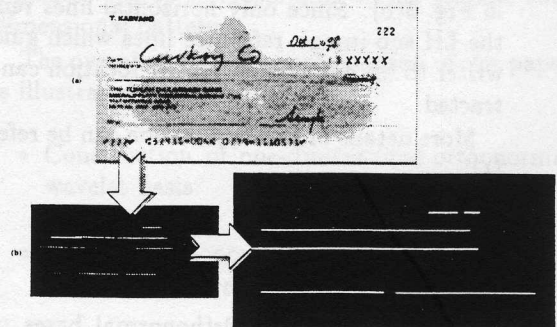


Figure 3: (a) A bank cheque, (b) its original and (c) enhanced LH sub-images

A document image can be transformed into four sub-images, namely:

- LL sub-image: both horizontal and vertical directions have low-frequencies. It corresponds to  $(V_j \otimes V_j)$ , and its orthonormal basis is  $\{\Phi_{j,k,m} | k, m \in \mathbb{Z}\}$ .
- LH sub-image: the horizontal direction has low-frequencies, and the vertical one has high-frequencies. It corresponds to  $(V_j \otimes W_j)$ , and its orthonormal basis is  $\{\Psi_{j,k,m}^1 | k, m \in \mathbb{Z}\}$ .
- HL sub-image: the horizontal direction has high-frequencies, and the vertical one has low-frequencies. It corresponds to  $(W_j \otimes V_j)$ , and its orthonormal basis is  $\{\Psi_{j,k,m}^2 | k, m \in \mathbb{Z}\}$ .
- HH sub-image: both horizontal and vertical directions have high-frequencies. It corresponds to  $(W_j \otimes W_j)$ , and its orthonormal basis is  $\{\Psi_{j,k,m}^3 | k, m \in \mathbb{Z}\}$ .

We are interested in the LH and HL sub-images. The LH sub-image results from a filter which allows lower frequency components to pass through along the horizontal direction, and higher frequencies along the vertical direction. That is an "enhancing" effect on the vertical, and "smoothing" effect on the horizontal. The result of the HL sub-image is opposite to that of the LH one. The higher frequencies can pass through along the horizontal direction, and lower frequency components along the vertical direction. That is an "enhancing" effect on the horizontal, and "smoothing" effect on the vertical.

For a Canadian Bank cheque shown in Fig. 3(a), there exists a grey-level background on it. To remove the grey-level background, the special properties of the sub-images can be used. Precisely, the LH sub-image of Fig. 3(a) has been obtained and shown in Fig. 3(b). Since only horizontal lines remain in the LH sub-image, reference lines which guides the writer to fill data in the proper location can be extracted.

More details of this application can be referred in [11].

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