Image Modeling by Hypergraph : Application to Noise Cancellation

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Abstract

In this paper, we introduce an image combinatorial model based on hypergraph theory. Hypergraph theory is an efficient formal frame for developing image processing applications such as restoration, segmentation and edge detection. We illustrate the effectiveness of such a model by a noise cancellation based on a basic combinatorial property associated with hypergraph. Experiments demonstrating the value of this approach are presented.

Key words - Noise reduction, neighborhood relations, hypergraph model, noise hyperedge, combinatorial model.

1 Introduction

Images are obtained as the result of the impression left by the light sent or reflected by objets of the world onto a photosensitive subfigure. One of the main challenges of disciplines such as visual psychophysics, computer vision, and robotics is to understand how, from the local properties of an image and in a way that is largely independent of image acquisition device, stable and reliable information about the world can be obtained.

In many fields of research, objects and interrelations are represented by graphs [1]. Proximity graphs are those in which points are fixed in n-dimensional space and adjacency is determined by the closeness of a pair of points relative to other points in the set. These graph are useful in solving problems in many areas of mathematics and computer science including computational geometry[7], geographic connectivity analysis, pattern analysis, artificial intelligence, and computer vision.

In this paper [1], we introduce an image neighborhood model built on hypergraph theory, which allows combinatorial [2] and geometrical approaches to image processing. In section 2 we begin with some preliminaries on graph and hypergraph theory.

2 Combinatoric background

Graphs are mathematical objects that can be used to model networks, data structures, data scheduling, computation, and a variety of other systems in which the relations between the objects in the system play a dominant role. Hypergraphs generalize the concept of a graph in order to cope with combinatorial problems. Our objective in this section is to introduce the terminology of graph and hypergraph theory and to define some familiar classes of graphs.

2.1 Elementary Graph Theoretic Definitions

This theory can be seen as a unified framework for many problems in different fields such as traffic networks, electrical circuits, and biology. It would be difficult to give a

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complete account of the area of graph problems.

A graph is a couple $G = (V, E)$, where V is a set of elements called vertices, and E is a set of unordered pairs of members of V called edges. Given a graph G, we denote by $\Gamma(x)$ the neighborhood of a vertex x, i.e., the set formed by all vertices adjacent to x :

$$\Gamma(x) = \{y \in V, \{x, y\} \in E\} \tag{1}$$

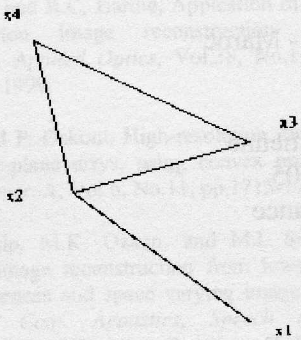

Figure 1: Example of graph G. In this graph the set of vertices is $V = \{x_1, x_2, x_3, x_4\}$, the set of edges is $\{E_1 = \{x_1, x_2\}, E_2 = \{x_2, x_3\}, E_3 = \{x_3, x_4\}, E_4 = \{x_2, x_4\}\}$, the neighborhood of x_2 est $\Gamma(x_2) = \{x_1, x_3, x_4\}$, and the degree of x_2 is $dx_2 = 3$.

2.2 Elementary Hypergraph Theoretic Definitions

The initial idea that gave rise to the hypergraph theory was extend certain classical results of graph theory. The initial idea that gave rise to the hypergraph theory was extend certain classical results of graph theory. Theoretical aspects of hypergraphs are presented [2].

2.2.1 Basic Concepts

An hypergraph H on a set X is a family $(E_i)_{i\in I}$ of nonempty subsets of X called hyperedges with

$$\bigcup_{i\in I} E_i = X \quad \forall i \in I \quad I = \{1, 2, ..., m\} \quad m \in N \tag{2}$$

Let us note $H = (X; (E_i)_{i\in I} = E)$

An hypergraph is often represented on the plane by points standing for the vertices; a hyperedge with cardinality 1 will be represented by a loop; an hyperedge with cardinality 2 will be represented by a line connecting its two elements, and an hyperedge with cardinality > 2 will be represented by a closed line surrounding its elements. Figure 2 is an example of the representation of an hypergraph.

Definition. An hyperedge is isolated [6] if and only if

$$\forall j \in I, j \neq i \;\; as \;\; E_i \cap E_j \neq \emptyset \;\; then \;\; E_j \subseteq E_i \tag{3}$$

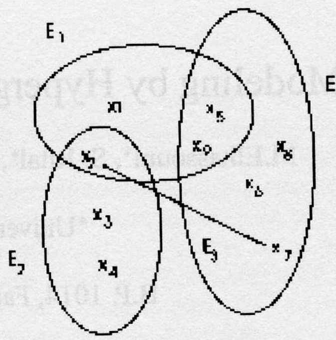

Figure 2: Representation of the hypergraph. In this example we have the set of vertices $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$, the set of hyperedges is $E_1 = \{x_1, x_2, x_5, x_9\}$, $E_2 = \{x_2, x_3, x_4\}$, $E_3 = \{x_2, x_7\}$, $E_4 = \{x_5, x_6, x_7, x_8, x_9\}$.

2.2.2 Neighborhood Hypergraph

Given a graph G, the hypergraph having the vertices of G as vertices and the neighborhood of these vertices as hyperedges (including these vertices) is called the neighborhood

$$H_G = (X, (E_X = x \cup \Gamma(x))) \tag{4}$$

hypergraph of G. To each graph we can associate a neighborhood hypergraph : The neighborhood hypergraph have been widely used in operational research and mathematical computation [9].

3 Image Processing and Neighborhood Relations

In this section some notions of graph theory applied to digital images are given. We introduce an alternative representation of the neighborhood relations which lets one define a new image neighborhood model.

3.1 Basic concepts

3.1.1 Image

A digital image is a two-dimensional discrete function that has been digitized both in spatial coordinates and in magnitude feature value. Throughout this paper a digital image will be represented by the application :

$$I : X \subseteq Z^2 \rightarrow Z^n \;\; with \;\; n \geq 1$$

where Z_n identifies the feature intensity level and X identifies a set of points called pixels in the image.

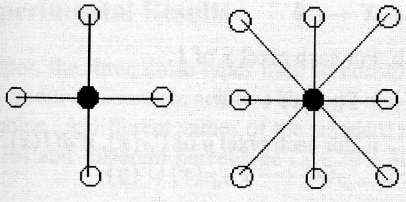

Figure 3: Neighborhood systems

3.1.2 Grids

In image processing we need a combinatorial topology since notions such as neighborhood and connectivity can be defined exactly by elements of a discrete space.

The most commonly used grids are 4-connected grids and the 8-connected grids defined on a square lattice of R^2. Figure 3 shows the neighbors of the lattice point in such a grid. The 4-neighborhood system can be formalized using the distance :

$$d_1(x,y) = |i-k| + |j-l| \tag{5}$$

where (i,j) and (k,l) denote, respectively, the coordinates of x and y on the lattice. The following relation gives the 4-neighbors of a point x on a lattice :

$$\forall x \in X, X \subseteq Z^2 : \Gamma(x) = \{y \in X, d_1(x,y) = 1\} \tag{6}$$

To define the 8-neighborhood system we use the following distance :

$$d_\infty(x,y) = max\{|i-k| + |j-l|\} \tag{7}$$

the following relation defines the 8-neighbors of a point x on lattice :

$$\forall x \in X, X \subseteq Z^2 : \Gamma(x) = \{y \in X, d_\infty(x,y) = 1\} \tag{8}$$

Let ρ be the minimum number of edges between any two vertices of a grid; we can call a neighborhood system of order n associated with the vertex x the set of vertices defined by

$$\Gamma_n(x) = \{y \in X, 1 \le \rho(x,y) \le n\} \tag{9}$$

Figure 4 shows the neighbors of order 2 of a lattice point.

3.2 Neighborhood Relations and Graphs

In image processing the notion of a neighborhood is a key concept. It is characterized by two essential properties :

- A pixel is not a neighbor of itself.
- If the pixel x is a neighbor of the pixel y, then y is a neighbor of x.

In other words, the neighborhood relation is an irreflexive and symmetric homogeneous relation that can be represented by a loopless graph.

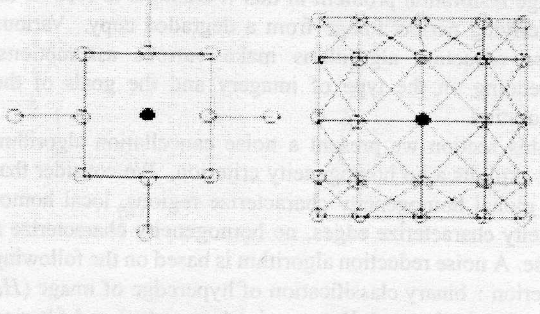

Figure 4: Neighborhood system at order 2

3.3 Neighborhood Relations and Hypergraph Model

In graph theory, the neighborhood notion is always built through an homogeneous relation. In the human visual system the perception of an object is dependent on its surroundings. The relations used in this process are built between the object and its neighborhood. Such a relation belongs to the class of heterogeneous relations.

Let $H = (X, E = (E_i)_{i \in I})$ be an hypergraph; the pairs (x, E_i) , where x belongs to the hyperedge E_i, define the heterogeneous relation. In image analysis an hypergraph associates a pixel x to its neighborhood, defined by a given neighborhood relation.

On a grid Γ_n, to each pixel x we can associate a neighborhood $\Gamma_{n,\alpha}(x)$, according to predicate α. The predicate α may be completely arbitrary provided it is useful for a task domain.

Formally an heterogeneous neighborhood relation $\Gamma_{n,\alpha}$ on an image I is a mapping on X with values in the power set $P(X)$ of X defined by $\Gamma_{n,\alpha}(x) = I^{-1}[I(x) - \alpha; I(x) + \alpha] \cap \Gamma_n(x)$ where x is a pixel, $I(x)$ is the feature intensity level of x, and $I^{-1}(B)$ is the reciprocal image of B by I such that :

$$I^{-1}(B) = \{x \in X, I(x) \in B\} \; with \; B \in P(X)$$

To this neighborhood we can associate an hypergraph defined by

$$H_{n,\alpha} = (X, (x \cup \Gamma_{n,\alpha}(x)))$$

This hypergraph will be called the image neighborhood hypergraph (INH). We suggest that such a general frame can provide a new basis for models [3] in different applications such as segmentation [4], noise reduction [5, 2] and edge detection.

4 An Application to Noise Cancellation

Noise affect digitized image in many ways, starting with the lens of the imaging hardware and ending at the digitalization of the captured image. Noise reduction is an image restoration problem in that it attempts to recover an underlying perfect image from a degraded copy. Various noise reduction algorithms make various assumptions, depending on the type of imagery and the goals of the restoration.

In this section we present a noise cancellation algorithm that exploits a no homogeneity criterion. We consider that the global homogeneity characterize regions, local homogeneity characterize edges, no homogeneity characterize a noise. A noise reduction algorithm is based on the following criterion : binary classification of hyperedge of image (H_0 noise hyperedge and H_1 no noise hyperedge) and filtering the noising parts. The noise model is defined by :

We say that $E_{\alpha,\beta}(x)$ is a noise hyperedge if it verifies one of the two conditions :

- the cardinality of $E_{\alpha,\beta}(x)$ is equal to one.
- $E_{\alpha,\beta}(x)$ is an isolated hyperedge and for every element y of open neighborhood on the grid of $E_{\alpha,\beta}(x)$, we have at least one $E_{\alpha,\beta}(y)$ isolated.

The lemma 1 shows that a noisy hyperedge must be isolated.

Lemma 1 If the cardinality of an hyperedge is equal to one, then this hyperedge is isolated.

Proof: given an hyperedge $E_{\alpha,\beta}(x)$ that the cardinality equal to 1. We suppose that it is not isolated. It is included in the hyperedge $E_{\alpha,\beta}(y)$, so $x \in \Gamma_{\alpha,\beta}(y)$ but $x \in \Gamma_{\alpha,\beta}(y)$ and the cardinality of $E_{\alpha,\beta}(x)$ is superior to 1. This brings us to a contradiction.

This definition is not sufficient, because an isolated hyperedge can belong to an edge element.

The strategy that we adopt is a binary/estimation strategy of decision. This process is like the one used in the decision theory and bayesian estimation. It differs nevertheless because the tested hypotheses and the prior knowledge used are of different natures. In bayesian strategy, they use distributions of the observation under each of the considered hypothesis. Classically processed problems concern the discrimination between a noise and a noisy signal. When the noise is detected, an algorithm of evaluation is used to approximate the signal. In this case the estimation is conditioned to the detection of the signal.

Our noise model permits to distinguish between a noise and any alternative hypothesis (contour, region). when a noisy zone is detected, we apply an estimation algorithm. The estimation is conditioned to the detection of the noise.

4.1 Noise cancellation algorithm

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1. Given
   Image $I$, real $\alpha$ and $\beta$
2. Construction of the neighborhood hypergraph $H_{\alpha,\beta}$ sur $I$
   a. $X \longleftarrow \emptyset$
   b. For each pixel $x$ of $I$
      i. $\Gamma_{\alpha,\beta}(x) \longleftarrow \emptyset$
      ii. for each pixel $y$ of $\Gamma_\beta(x)$, if $d(I(x), I(y)) \leq \alpha$ then $\Gamma_{\alpha,\beta}(x) \longleftarrow \Gamma_{\alpha,\beta}(x) \cup \{y\}$
      iii. $X \longleftarrow X \cup \{x\}$
      iv. $E_{\alpha,\beta}(x) \longleftarrow \{\Gamma_{\alpha,\beta}(x) \cup \{x\}\}$
      v. $H_{\alpha,\beta}(x) \longleftarrow (X, (E_{\alpha,\beta}(x))_{x \in X})$
3. Determination of isolated hyeregdges of $H_{\alpha,\beta}$
   For each vertex $x$ of $X$
      i. $E' \longleftarrow \emptyset$
      ii. $E' \longleftarrow \bigcup_{y \in E_{\alpha,\beta}(x)} E_{\alpha,\beta}(y)$
      iii. if $E' = E_{\alpha,\beta}(x)$ then $E_{\alpha,\beta}(x)$ is an isolated hyperedge.
      iv. $E^{is}_{\alpha,\beta}(x) = E_{\alpha,\beta}(x)$
4. Detection of hyperedges of noise
   For every $E^{is}_{\alpha,\beta}(x)$
      i. If the cardinality of $E^{is}_{\alpha,\beta}(x)$ is equal to one, then the class this hyperedge like an hyperedge like an hyperedge of noise. $E^b_{\alpha,\beta}(x) = E_{\alpha,\beta}$;
      ii. If for all pixel y of $\Gamma(E^{is}_{\alpha,\beta}(x))$, there is in minimum one $E_{\alpha,\beta}(x)$ isolated then class $E_{\alpha,\beta}(x)$ like an hyperedge of noise
      iii. $E^b_{\alpha,\beta}(x) = E_{\alpha,\beta}$.
5. Estimation
   For each hyperedge of noise $E^b_{\alpha,\beta}(x)$ Replace the intensity of $E^b_{\alpha,\beta}(x)$ by the value of a functionnal dependent of intensity of $\Gamma(E^b_{\alpha,\beta}(x))$.
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4.2 Properties of the Algorithm

Lemma2 (of convergence, complexity and singleness) For α and β given the algorithm converges to an unique solution. His complexity is in $O(n)$ (n designing the size of image).

Proof. A study of steps of the algorithm permits to note that the complexity of each of these is raised by the number of pixels of the Image to a multiplicative coefficient. This coefficient is the number of restrained pixels in hyperedge, raised himself by the cardinality maximum of hyperedges. Therefore, the complexity of the algorithm is in $O(n)$. For α and β fixed, it exists a single hypergraph $H_{\alpha,\beta}$ associated to an image data, where the convergence toward an unique classification and an unique solution.

5 Experimental Results

In this paper, the three noise types used to corrupt the images were additive Gaussian, additive Uniform and "salt and pepper" noise. A different values of the standard deviation (5,10,20,30) and different percentage (5%,20%,30 %) are used.
To qualify the model, it is necessary to evaluate detection and false alarm probabilities. We led a serial of experimentation in order to estimate the operational curves of decision in the case of perturbation that have additive nature, and the one of the Impultional. The operational curves has been estimated using the method of simulation of Monte Carlo [11].

5.1 Model performance

The main information's that one can extract from the experimentation are :

- More the variance of the noise is raised more the model is capable to discern hyperedges of noise. The figure 5 represents the operational curves obtained for an additive perturbation for 3 values of the standard deviation (10, 20, 30) that affects 30% of pixels of the image (image Lena).

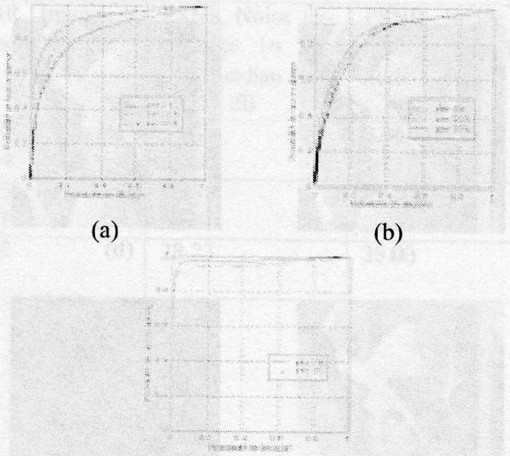

(a) (b)

Figure 5: Evolution of the algorithm according to the standard deviation, (a) gaussian additive noise with 30 % of noisy pixel , (b) uniform additive noise with 30 % of noisy pixels

- Performances decrease when the percentage of noisy pixels increases. Indeed, the spatial homogeneity of noisy data grows and the model has difficulty to discern noise hyperedges (see figure 6);
- The model is more effective when the perturbation is additive. For the same noise distribution, the model is more capable to detect it when this perturbation is of additive nature rather than multiplicative one.

5.2 Results evaluation

After tracing the operational curves that give an idea for extracting the optimal threshold α [13], we applied a median filter to a noisy hyperedge of natural images (Lena (figure 7), house (figure 8), parrots (figure 9)) corrupted by the three noise types used (additive Gaussian, additive Uniform, 'salt and pepper'). Peak SNR (PSNR) is used to give a quantitative evaluation of the filtering effects. In this paper the optimum threshold values for many images that have a maximal decision probabilities and minimal false alarm is contained in the interval 2, 5, for our application we have used a global threshold equal to 2. We present those data in table 1 for different percentages and variances of noise, and we show the filtering results. In most cases, the performance of the proposed algorithm is close to the optimal one.


(a) (b)

(c)

Figure 6: Evolution of the algorithm according to injected noise percentage in the image (a) additive Gaussian noise with a value of $\sigma = 30$, (b) additive Uniform noise with a value of $\sigma = 30$, (c) salt and pepper noise

For the processed images, we have filtered the noise hyperedges and compared with the results obtained with SUSAN algorithm [12]. If we choose a visual quality as a criterion, we observe a difference in restitution of edge, which proves the superiority of our approach.
If we compare our approach to the classical procedure of noise cancellation by filtering (mean, median, Multilevel median [10] ...), the fact of applying the estimation step only for the detected noise inevitably to a good edge preserving.

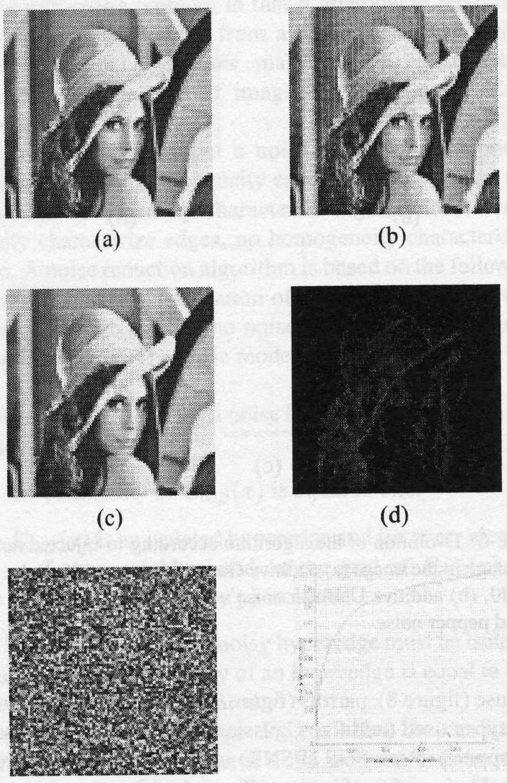
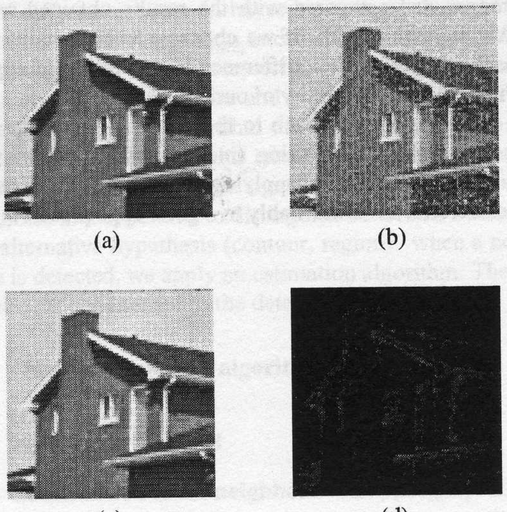

(a) (b)
(c) (d)
(e) (f)

Figure 7: (a) Original image of Lena, (b) image of Lena corrupted by 30 % additive Gaussian noise, and PSNR = 24,18 dB, (c) image results ($\alpha = 2, \beta = 1$) by filtering hyperedges of noise by median 3x3 filter, PSNR=29,56dB, (d) image error, (e) additive Gaussian noise N(0,30), (f) operational curve of detection.


(a) (b)
(c) (d)

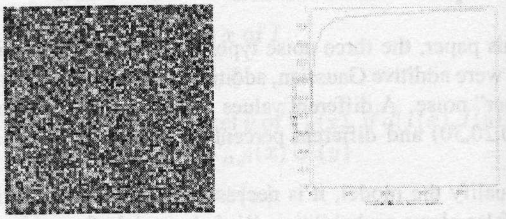

(e) (f)

Figure 8: (a)Original image of house, (b) image house corrupted by an additive Uniform noise PSNR = 17.14 dB, (c)image results ($\alpha = 2, \beta = 1$) by filtring hyperedges of noise by median 3x3 filter, PSNR = 30,58 dB, (d) image error, (e) additive Uniform noise 30 %, (f) operational curve of detection

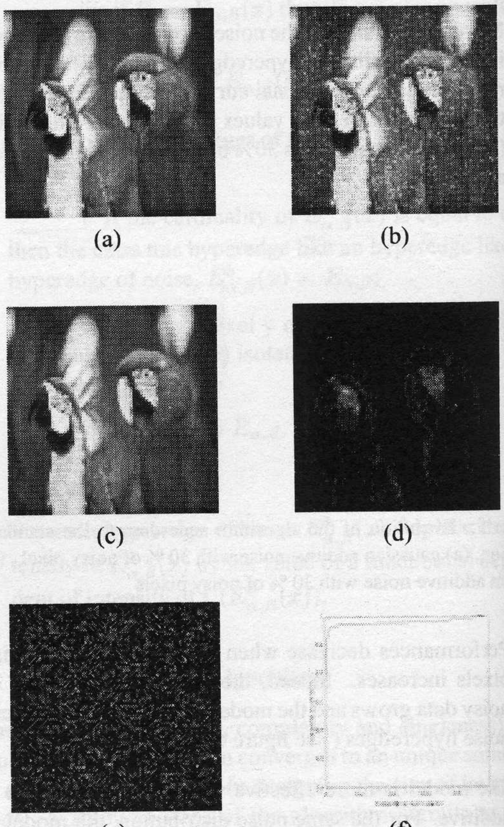

(a) (b)
(c) (d)
(e) (f)

Figure 9: (a) Original image of parrots, (b) image of parrot corrupted Impultionnal noise PSNR = 17,72 dB, (c) image results ($\alpha = 2, \beta = 1$) by filtering hyperedges of noise by median 3x3 filter, PSNR=28,03 dB, (d) image error, (e) Impultionnal noise 5%, (f) operational curve of detection

Image 256X256	Nature of Noise	Standard Deviation with mean = 0	Percentage of noise in %	PSNR Image + Noise in dB	PSNR Noise hyperedge ($\alpha = 5$) + median filter 3x3 in dB	PSNR Image + median filter 3x3 in dB
house	Additive Uniform	20 20	20 30	21.62 17.14	33.43 30.60	32.64 30.10
Lena	Additive Gaussian	20 30	30 30	27.53 24.18	30.38 29.71	29.97 29.55
Parrots	Salt and Pepper	-	5	17.72	28.23	28.03

Table 1: Comparative filtering results in PSNR for three natural images (Lena, house, parrots) corrupted by the three Noise types(additive Uniform, additive Gaussian and ”salt and pepper”), and the PSNR of filtered image by (filtering only the hyperedges of noise) and in the hole image

6 Conclusion

This paper has described a new approach of noise cancellation using the neighborhood hypergraph model based on the mathematical background of combinatorics. This method gives a new definition to characterize the noise hyperedges. We evaluated the performance of our model with quantitative values. Experimental results indicate that our approach is superior to the other methods tested in noise detection and reduction.

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