Retrieval of the Calibration Matrix from the 3-D Projective Camera Model

Gamal H. Seedahmed* and Toni Schenk
Photogrammetry Group
Dept. of Civil and Environmental Engineering and Geodetic Science
The Ohio State University
2070 Neil Avenue, Columbus OH 43210-1275 USA
seedahmed.1@osu.edu
schenk.2@osu.edu

Abstract

By relating the projective camera model to the perspective one, the intrinsic camera parameters give rise to what is called the calibration matrix. This paper presents two new methods to retrieve the calibration matrix from the projective camera model. In both methods, a collective approach was adopted, using matrix representation. The calibration matrix was retrieved from a quadratic matrix term. The two methods were framed around a correct utilization of Cholesky factorization to decompose the quadratic matrix term. The first method used an iterative Cholesky factorization to retrieve the calibration matrix from the quadratic matrix term. The second method used Cholesky factorization to factor the quadratic matrix term but after its inversion. The basic argument behind the two methods is that: the direct use of Cholesky factorization does not reveal the correct decomposition due to the missing matrix structure in terms of lower-upper ordering. This study presents two new algorithms to rebuild the missing matrix structure. In both methods, a successful retrieval of the calibration matrix was achieved. This paper explains the key ideas behind the two methods, accommodated with a simulated example to demonstrate their validity.

1 Introduction

In this study the term calibration will be reserved for the intrinsic camera calibration. Calibration of cameras, analog and digital-alike, is a prerequisite task for the precise extraction of metric information from imagery in photogrammetry, computer vision, and other vision applications in which the precise quantitative measurements are needed.

Most current vision applications, employed commercial off-the-shelf (COTS) cameras that exhibit a considerable amount of distortions due to various reasons. The camera assembly is often misaligned, the CCD chip may not be orthogonal to the optical axis, the effective focal length may not be known, and the camera lens may exhibits a high radial distortions. The removal of these distortions constitutes the objectives of geometric camera calibration, see [1]. Generally, camera calibration is formulated under the perspective or the projective camera model. Under the perspective camera model, a full calibration model, which retains the geometric integrity of the extracted features, can be achieved at the cost of non-linear system of equations. On the other hand, a partial calibration model can be achieved under the projective camera model but with the main advantage of having a closed form solution.

By establishing the relationship between the projective and perspective camera models, the calibration parameters can be retrieved either collectively or term-wise. The collective retrieval gives rise to what is called the calibration matrix. On the contrary, there exist a term wise retrieval for the calibration matrix. In the term-wise retrieval, the projective camera parameters are written as a function of the perspective camera model, which give rise to a set of equations that have to be solved sequentially to recover the calibration parameters, see [2].

Cholesky factorization in its original format is suggested as a decomposition method to retrieve the calibration matrix from the projective camera model, see [3] and [4]. In the subsequent sections of this paper, it will be shown that this is not a general decomposition. It is interesting to mention that the 3-D projective camera model is interpreted as perspective transformation combined with a 2-D affine transformation, see [2]. The inner meaning of this interpretation is that the image coordinates do not necessary to be referenced to the principal point. In other words, we can choose any image coordinate system and still be able to retrieve the correct perspective camera model. Clearly, this is not the case when using Cholesky factorization in its original format. We showed in the sequel of this paper that Cholesky factorization is not a valid decomposition when the

* The author is also with The Pacific Northwest National Lab.
E-mail: Gamal.Seedahmed@pnl.gov

principal point is displaced from its true location or the image coordinates measurements are not based on the center of perspectivity.

In this paper, we present two new methods to retrieve the calibration matrix that belong to the class of collective retrieval. Both methods used Cholesky factorization in the correct sense. The first method used an iterative Cholesky factorization to retrieve the calibration matrix from a quadratic matrix term. The second method used Cholesky factorization to factor the quadratic matrix term but after its inversion. The key idea behind the two methods is that: the direct use of Cholesky factorization will not reveal the correct decomposition for the calibration matrix housed in the quadratic matrix term, despite the fact that we have a symmetric positive definite matrix, and this is due to the missing matrix structure in terms of lower-upper ordering. The quadratic matrix term, which housed the calibration matrix, has an upper-lower ordering. The two methods rebuild the missing matrix structure for Cholesky factorization and enable the correct retrieval of the calibration matrix.

In the context of the perspective camera model, the calibration matrix serves three distinct purposes: first, the establishment of the central projection, second, recovering the aspect ratio between the two axes, and third, maintaining the orthogonality between rows and columns of the CCD chip. The developed methods serve as practical algorithms to establish the perspective camera model in the context of projective one.

This paper is organized as follows. Section 2 briefly reviews the 3-D projective camera model and emphasizes its functional linearity. Section 3 presents the principle of the matrix factorization since it explains the relationship between the projective and the perspective camera model collectively. Section 4 and 5 present the new methods of the calibration matrix retrieval and they are explaining the key ideas behind them. Section 6 presents the experimental results and the discussions. Finally, section 7 concludes the paper.

2 The 3-D Projective Camera Model

In the projective model the camera is considered as a system that performs a linear projective transformation from the projective space P^3 into the projective plane P^2. Mathematically this mapping can be written as:

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} L_1 & L_2 & L_3 & L_4 \\ L_5 & L_6 & L_7 & L_8 \\ L_9 & L_{10} & L_{11} & L_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \qquad (1)$$

where

x, y image coordinates.

X, Y, Z Object space coordinates.

$L_1..L_{12}$ camera parameters.

λ the scale factor.

T remove the scale factor, λ, we can write the following relations:

$$x = \frac{L_1X + L_2Y + L_3Z + L_4}{L_9X + L_{10}Y + L_{11}Z + L_{12}} + e_x \qquad (2)$$

$$y = \frac{L_5X + L_6Y + L_7Z + L_8}{L_9X + L_{10}Y + L_{11}Z + L_{12}} + e_y \qquad (3)$$

Equations (2) and (3) are the non-linear version of the 3-D projective model. The projective transformation leads to a set of homogenous equations defined up to a scale factor. The basic idea to remove the scale ambiguity is to constrain the solution either by a partial norm or by full norm of the unknown parameters. In this study, $L_{12}=1$, is chosen as a scaling criterion, see [5]. Clearly, this is not the optimal choice but it will satisfy the purpose of this study. A linear version of equations 2 and 3 can be written as:

$$\begin{aligned} x = {} & XL_1 + YL_2 + ZL_3 + L_4 - (x-e_x)XL_9 - \\ & (x-e_x)YL_{10} - (x-e_x)ZL_{11} + e_x \end{aligned} \qquad (4)$$

$$\begin{aligned} y = {} & XL_5 + YL_6 + ZL_7 + L_8 - (y-e_y)XL_9 - \\ & (y-e_y)YL_{10} - (y-e_y)ZL_{11} + e_y \end{aligned} \qquad (5)$$

where

e_x *and* e_y are the true unknown errors associated with the image coordinates.

X, Y, *and* Z *are* treated as error free coordinates in equations (4) and (5).

It is important to note that the stochastic modeling of equations (4) and (5) leads to a non- linear mathematical model, which requires an iterative solution. In fact, this is a stochastic non-linearity and not a functional one. Very good approximations can be obtained by neglecting the stochastic component at the first iteration and then proceed to the non-linear solution, which models the randomness depicted in equations (3) and (4). Least squares error modeling is used as optimization criterion.

3 Principles of Matrix Factorization

This section reviews the principles of matrix factorization. The matrix factorization technique provides a compact link between the projective and the perspective camera models. The perspective camera model can be written as a product of the following matrices using homogenous coordinates:

$$x = KR[I \mid -X_o]X \tag{6}$$

$$X_o = \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} \tag{7}$$

where

x : image coordinates vector.
K : calibration matrix.
R : rotation matrix.
X_o : the position of the camera in the object space.
X : coordinates vector of a point in the object space.
I : the identity matrix.

The general form of the calibration matrix is:

$$K = \begin{bmatrix} C_x & \alpha & x_p \\ 0 & C_y & y_p \\ 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Where:

x_p, y_p are the coordinates of the principal point.
C_x, C_y focal length along the x and y axes.
α : skewness factor.

From equations (1) and (6) the following equivalence between the projective and perspective cameras can be inferred:

$$\begin{bmatrix} L_1 & L_2 & L_3 & L_4 \\ L_5 & L_6 & L_7 & L_8 \\ L_9 & L_{10} & L_{11} & L_{12} \end{bmatrix} = KR[I \mid -X_o] \tag{9}$$

From equation (9) we can write:

$$KR = D \tag{10}$$

where:

$$D = \begin{bmatrix} L_1 & L_2 & L_3 \\ L_5 & L_6 & L_7 \\ L_9 & L_{10} & L_{11} \end{bmatrix} \tag{11}$$

Also from equation (9) we can write:

$$-KRX_o = d \tag{12}$$

Where:

$$d = \begin{bmatrix} L_4 \\ L_8 \\ L_{12} \end{bmatrix} \tag{13}$$

From equation (10) we can write a quadratic term for the calibration matrix as follows:

$$(KR)(KR)^T = DD^T \tag{14}$$

Then:

$$KK^T = DD^T \tag{15}$$

Since:

$$RR^T = I \tag{16}$$

Where:

I : is the identity matrix.

The normalized calibration matrix can represented by:

$$K = \frac{1}{K_{33}} \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ 0 & K_{22} & K_{23} \\ 0 & 0 & K_{33} \end{bmatrix} \tag{17}$$

Equation (15) is the starting point for the two methods presented in this paper. The basic argument behind the two methods is that: the direct use of Cholesky factorization will not reveal the correct decomposition to the matrix (DD^T), despite the fact that we have a symmetric positive definite matrix, and this is due to the missing matrix structure in terms of lower-upper ordering. The missing matrix structure can be confirmed by checking the structure of the calibration matrix *(K)* in connection with the matrix (DD^T).

4 The First Method

The first method utilized the Cholesky factorization coupled with an iterative update of the principal point assuming that the skewness factor is very small. At every iteration the principal point was updated and the observed image coordinates were corrected due to the principal point displacement. We observed that after a very few iterations, the solution converged to the correct calibration parameters .

The key ideas behind this method can be captured by the following two arguments. First, Cholesky factorization alone will not reveal the correct decomposition of the matrix (DD^T) since we had upper-lower matrix structure instead of lower-upper matrix structure. Second, the iterative solution reduced the factored matrix to a diagonal structure, which made Cholesky factorization a valid decomposition. Step-wise this method can be stated as follows:

1. Initialize the principal point to zero.
2. Compute the camera parameter using equations (4) and (5).
3. Form the quadratic matrix term, (DD^T), using equation (15).
4. Apply Cholesky factorization to the quadratic matrix. This step reveals the un-normalized calibration matrix *(K)*.
5. Normalize the calibration matrix by dividing its elements by *K(3,3)* using equation (17).
6. Extract the principal point from the normalized calibration matrix.
7. Update the principal point.
8. Displace the observed image coordinates using the updated principal point.
9. Repeat steps 2-8 until the convergence of the solution to a stable principal point.

The net result of this algorithm is a reduced calibration matrix in the sense that the elements correspond to x_p and y_p are equal to zero. The principal point solution is recovered in two separate terms.

5 The Second Method

The second method is based on a very simple idea. This idea states that: by inverting the matrix (DD^T) we will end-up with the correct order in terms of lower-upper matrix structure, which will lend itself to a direct Cholesky factorization and by inverting the result of factorization we will end-up with the correct calibration matrix. Step-wise this method can be stated as follows:

1. Compute the camera parameters using equations (4) and (5).
2. Form the quadratic matrix term, (DD^T), using equation (15).
3. Invert the matrix (DD^T).
4. Find the Cholesky factorization of the matrix $(DD^T)^{-1}$.
5. Invert the factored matrix and this is represents the un-normalized calibration matrix *(K)*.
6. Normalize the calibration matrix, by dividing its elements by *K(3,3)*, to end-up with the calibration matrix *(K)* using equation (17).

The net result of this algorithm is the full calibration matrix as depicted by equation (17). This method does not require an iterative solution compared to the previous on.

6 Experimental Results

This section presents the experimental results of a simulated example using a single image. We set-up 8 control points at the object space, see table 1, and project them to the image space using the perspective camera model with specified extrinsic camera parameters presented in table 3. Table 4 shows the intrinsic camera parameters, used in connection with the extrinsic parameters, to project the control point into the image space, see table 2. This example represents a typical aerial photography in photogrammetry.

Table 1: Object space Points in meters.

POINT ID	X	Y	Z
P_1	-200.0	-200.0	100.0
P_2	-200.0	2200.0	100.0
P_3	2200.0	2200.0	100.0
P_4	2200.0	-200.0	100.0
P_5	2200.0	1000.0	100.0
P_6	200.0	1000.0	100.0
P_7	900.0	2000.0	50.0
P_8	1100.0	100.0	150.0

Table 2: Image space points in mm.

POINT ID	X	Y
P_1	-96.9105	-90.3249
P_2	-81.8805	95.4951
P_3	105.4855	85.5611
P_4	94.1925	-106.8049
P_5	99.9025	-9.5429
P_6	-58.6375	1.4631
P_7	1.4425	73.6151
P_8	6.7605	-76.9579

Table 3: Extrinsic camera parameters.

Xo (m)	Yo (m)	Zo (m)
1000.1	999.81	2000.1
ω^o	φ^o	κ^o
1.0002	1.5	3.9999

where

ω, φ, *and* κ: are the elements of the rotation matrix *R*.

In this study we design four experiments to study the validity of the two methods. The only difference from experiment to experiment is that the image coordinates are shifted from their true locations using four different sets of principal point as depicted in table 4. In all experiments we used an identical skewness factor equal to 0.13615.

Table 4: Intrinsic camera parameters.

Experiment#	C_x mm	C_y mm	x_p mm	y_p mm
1	150.01	149.91	0.1	0.2
2	150.01	149.91	0.130	5.4
3	150.01	149.91	9.01	11.97
4	150.01	149.91	19.01	21.97

In the sequel of this section, we used experiment #1 and #4 to demonstrate the validity of the two methods. Using the Cholesky factorization in its original format and applying it to experiments #1 and #4, we end-up with the following calibration matrices respectively:

$$K(\#1) = \begin{bmatrix} 150.0122 & 0.1363 & 0.1 \\ 0. & 149.9069 & 0.2 \\ 0. & 0. & 1. \end{bmatrix}$$

$$K(\#4) = \begin{bmatrix} 154.0212 & 2.9509 & 19.7230 \\ 0. & 154.2948 & 22.4161 \\ 0. & 0. & 1. \end{bmatrix}$$

It is evident that the direct use of Cholesky factorization alone will not reveal the correct calibration matrix. By examing, the elements of the K matrix, in experiments #1 and #4, we can conclude that the use of Cholesky factorization in its original format will lead to incorrect calibration matrix, even when the principal point displacement is very small.

By using the first method, we are able to retrieve the correct calibration parameters. By examining the graph depicted in fig. 1, we can deduce that we need 2 to 3 iterations to obtain the correct solution of the reduced calibration matrix and the principal point.

The reduced calibration matrix is:

$$K = \begin{bmatrix} 150.01 & 0.13615 & 0.0 \\ 0. & 149.91 & 0.0 \\ 0. & 0. & 1. \end{bmatrix}$$

The principal point solution is:

$x_P = 19.01$

$y_P = 21.97$

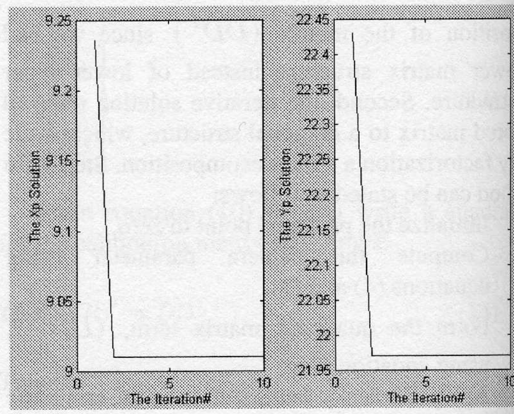

Figure 1: Principal Point Solution vs. Iteration#

By applying the second method to the quadratic matrix term, we end-up with the correct calibration matrix K.

$$K = \begin{bmatrix} 150.01 & 0.13615 & 19.01 \\ 0. & 149.91 & 21.97 \\ 0. & 0. & 1. \end{bmatrix}$$

7 Conclusions

- The basic idea behind the two methods is that: the direct use of Cholesky factorization will not reveal the correct decomposition, despite the fact that we have a symmetric positive definite matrix, and this is due to the missing of the lower-upper matrix structure.
- The two methods rebuild the missing matrix structure and enable the correct retrieval of the calibration matrix.
- The first method adopts an iterative strategy that led to the correct retrieval of the calibration matrix.
- The second method avoid the iterative strategy by applying Cholesky factorization in the inverse domain and this reveal the correct matrix structure in term of lower-upper structure, which led also to the correct retrieval of the calibration matrix.
- The Cholesky factorization is a valid method if the principal point displacement is very small.
- The presented methods preserve the original interpretation of the 3-D projective camera model as a perspective transformation combined with a 2-D affine transformation. Clearly, this interpretation give us the freedom to choose any image coordinate system to reference the image coordinates and still be able recover the correct calibration parameters.

- It is not clear why the first method converge to the correct solution. This issue will be investigated in a later study.
- For practical applications, we can use the second method, which does not require an iterative solution.
- Matrix factorization provides a compact link between the projective and the perspective camera models.

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