

Geometrical and Topological Informations for Image Segmentation with Monte Carlo Markov Chain Implementation

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Abstract

Image segmentation methods based on Markovian assumption consist in optimizing a Gibbs energy function which depends on the observation field and the segmented field. This energy function can be represented as a sum of potentials defined on cliques which are subsets of the grid of sites. The Potts model is the most commonly used to represent the segmented field. However, this model only expresses a potential on classes for nearest neighbor pixels. In this paper, we propose the integration of global informations, like the size of a region, in the local potentials of the Gibbs energy. To extract these informations, we use a representation model well known in geometric modeling: the topological map. Results on synthetic and natural images are provided, showing improvements in the obtained segmented fields.

1 Introduction

The main objective of image segmentation methods is to find areas of homogeneous pixels. In such context, there are two main research axes: the boundary and the region based methods [1]. Fusions of both approaches have also been proposed. Nevertheless, either you are using one approach or the other, a partitioned image is the result of the segmentation operation. This partition is composed of subsets called "classes" for a set of homogeneous pixels, and "regions" for a set of homogeneous and related pixels.

The interest in such treatment can be found in many applications like image compression [2], biomedical image analysis [3, 4],...

For two decades, Monte-Carlo Markov Chain (MCMC) methods have received a increasing interest [5, 6]. This is due to a rigorous mathematical background and the growing power of computers. Since the works of Geman & Geman on image restoration [7], these methods have been applied successfully to image segmentation using different implementations, *i.e.* using stochastic or deterministic algorithms [8, 9].

In such approaches, the image is considered as a hierarchical field composed by an observation field and a segmented or label field. Various parametric models have been proposed to represent the textures of the observation field [10, 11]. But in most cases, the commonly used Potts model [12] is proposed to represent the label field. This model only takes into account the label values of the nearest neighbors in the Gibbs energy function. Geman & Geman [7] tried to add more informations to the Potts model using boundaries estimated during the optimization process. Some boundary configurations were penalized. In [13], this idea has been extensively used. A drawback can be found to this approach in the sense that *a priori* weighted values were given to each boundary configurations.

In this paper, we propose to add informations about the label field's geometry and topology to the energy function. These informations, such as the form or size of a region, would be considered as "global", as opposed to "local" informations provided by the Potts model. Of course, the Potts model will be still used because of its regularization effect, but potentials functions taking into account "global" informations will be added to the Gibbs energy function.

To get these additional informations, we propose the use of a combinatorial model: the topological map [14]. This structure encodes an efficient way all topological and geometrical informations contained in the image. In a sense, topology can be seen as the representation of structural relationships between regions, for instance adjacency. These relationships can be used to characterize a region and, furthermore, a set of regions.

Our segmentation algorithm consists in two steps and is unsupervised considering the number of classes and the model parameters. A first step allows to estimate the number of classes and the model parameters of a Gaussian mixture using a Stochastic Expectation Maximization (SEM) algorithm [15, 16]. During the second step, a Simulated Annealing (SA) based method is applied in order to obtain the final label field. This is the step in which we will introduce topological or geometrical informations extracted from the topological map.

In the second part, we start with a summary on the principles of Markovian segmentation. Then, after having explained in the third part the operation of topological maps, we will propose in the fourth part an integration model to include these maps in a Markovian segmentation algorithm. We will also give results on synthetic and natural images and perspectives following this work.

2 Markovian segmentation algorithm

2.1 Definitions

Let $X = \{X_s, s \in S\}$ a family of random variables on a regular grid $S = \{s_1, s_2, \dots, s_N\}$ which is a finite subset of $\mathbb{Z}$ and an image $x = \{x_s, s \in S\}$, realization of X . Each X_s has the same state space that can be $\Omega_x = \{0, 1, \dots, 255\}$, the set of gray pixel values, or $\Omega_x = \mathbb{R}, \dots$. We now suppose that there is another random field $L = \{L_s, s \in S\}$ called the label field with state space $\Omega_l = \{c_1, c_2, \dots, c_K\}$ for which a realization will be $l = \{l_s, s \in S\}$. In our case, a label represents the class to which the site belongs.

From this definition, we suppose that X and L are defined in a hierarchical way : for each l_s , we have a conditional model for X_s that can be, for example, an AR model or a Gaussian distribution [8, 11].

The likelihood method defines the *a posteriori* law $P(L = l/X = x)$ as the probability to get one specific realization of the label field knowing the observation field:

$$P(L = l/X = x) = \frac{P(X = x, L = l)}{P(X = x)} \quad (1)$$

Using the Bayes formula, we can also write $P(L = l/X = x)$ in function of the *a priori* law $P(L = l)$:

$$P(L = l/X = x) = \frac{P(X=x/L=l)P(L=l)}{P(X=x)} \quad (2)$$

$$\propto P(X = x/L = l)P(L = l)$$

since we consider the probability of the observation field $P(X = x)$ as a constant value.

From now, we will simply write as follows:

$$P(l/x) \propto P(x/l)P(l) \quad (3)$$

Maximizing this *a posteriori* law (MAP) is equivalent to getting the most probable realization of the label field, knowing the observation field.

2.2 Markov field and Gibbs distribution

We call $\mathcal{V} = \{v_s\}$ a neighboring site system for S . A neighboring system has the two following properties:

$$\begin{aligned} (1) & s \notin v_s \\ (2) & s \in v_r \Leftrightarrow r \in v_s \end{aligned} \quad (4)$$

A subset $c \subseteq S$ is a clique if every pair of distinct sites in c are neighbors and $\mathcal{C}$ is the set of cliques. In our study, we will only use the neighboring system based on the four nearest neighbors (4-connectivity) and the eight nearest neighbors (8-connectivity).

A random field $Y = \{Y_s, s \in S\}$ is a Markovian field associated to the system v_s if and only if:

$$\begin{aligned} P(Y = y) & \triangleq P(y) > 0, \forall y \in \Omega_y \\ P(Y_s/Y_r, r \in S - \{s\}) & = P(Y_s/Y_r, r \in v_s) \end{aligned} \quad (5)$$

From Hammersley-Clifford's theorem, the distribution function of a Markov random field follows a Gibbs' law:

$$P(y) = \frac{\exp(-U(y))}{Z} \quad (6)$$

in which U is an "energy function" and $Z = \sum_y \exp(-U(y))$ a normalizing factor. U can be written as:

$$U(y) = \sum_{c \in \mathcal{C}} \Phi_c(y) \quad (7)$$

where $\Phi_c : \Omega_y^{|c|} \rightarrow \mathbb{R}$ is the potential function with $\Phi_c(y) = \Phi_c(y_c)$. $|c|$ is the cardinality of c and in the special case of a clique, is called its order. y_c denotes the set $\{y/y \in c\}$.

Therefore, if the Markovian assumption is made about the (X, L) hierarchical field of part 2.1, (3) can be written in a general way as follows:

$$P(l/x) \propto \frac{\exp(-U_1(x, l) - U_2(l))}{Z(x)} \quad (8)$$

in which U_1 is an "energy function", depending on the conditional observation fields, and U_2 another energy function depending on the label field. $Z(x)$ is the normalizing factor depending on x . Let define now $U(l) = U_1(x, l) + U_2(l)$ as x is the given observation field.

The MAP estimation of l can now be written using (3) and (8):

$$\hat{l} = \arg \max_{l \in \Omega_l} [P(l/x)] = \arg \min_{l \in \Omega_l} [U_1(x, l) + U_2(l)] \quad (9)$$

We have to optimize $U : \Omega_l^{|S|} \rightarrow \mathbb{R}$. SA method based on Gibbs sampler has shown to be appropriate to such an optimization problem. In next part, we present the stochastic model we have chosen, and the form of the corresponding potentials.

2.3 Stochastic modeling

The conditional field X is now supposed to be composed of K textures. Hence, we have K probability laws describing the gray pixel values in x for the different regions belonging to the K textures. Different choices can be done for

these probability laws [11]. In this paper, we are making the assumption that the observation field is a mixture of K independently and identically distributed (i. i. d.) Gaussian processes. Although such assumption may sound simplistic, it happens to have shown good results on a great number of images. We decided to use this well-known model in order to focus on the topological maps integration, it may be replaced in the future by a model that fits better to the image. Besides, using this model also allows us to use a Maximum Likelihood-based algorithm such as SEM [15] to estimate K and the model parameters, $\theta_k = \{\mu_k, \sigma_k\}$, $k = 1 \dots K$, where μ_k and σ_k are respectively the average and the standard deviation of the texture k .

We suppose that K and $\theta = \{\theta_k, k = 1 \dots K\}$ are known parameters. Then, we will have:

$$P(x/l) = \prod_{s \in S} f(x_s/l_s) \quad (10)$$

with $f(x_s/l_s)$ a Gaussian law $\mathcal{N}(\mu_{l_s}, \sigma_{l_s})$. We can write $U_1(x, l)$ as follows:

$$U_1(x, l) = \sum_{s \in S} \left(\frac{(x_s - \mu_{l_s})^2}{\sigma_{l_s}^2} + \log(\sigma_{l_s}) \right) \quad (11)$$

This energy can be seen as a sum of local potentials noted Φ^1 based on first order cliques:

$$\Phi^1(x_s, l_s) = \frac{(x_s - \mu_{l_s})^2}{2\sigma_{l_s}^2} + \log(\sigma_{l_s}) \quad (12)$$

It remains to propose a model for the label field. In many papers, the Potts model is used [12]. This assumption corresponds to a general hypothesis about the label field: the different classes can be permuted without any influence on the probability law. We propose the use of a more general formalism, writing Potts' model, as it is used in [8], back into its general form, the Besag's auto-binomial model [17]:

$$U_2(l) = \sum_{s \in S} \Phi(l_s) + \beta \sum_{\langle i, j \rangle_1} (1 - \delta(l_i, l_j)) + \frac{\beta}{\sqrt{2}} \sum_{\langle i, j \rangle_2} (1 - \delta(l_i, l_j)) \quad (13)$$

with δ , the Kronecker symbol, β , the weight term of the Potts model and p , the clique's order, $p = 1, 2$:

$$\begin{aligned} \langle i, j \rangle &\in S, i \neq j : \\ \langle i, j \rangle_p &\Leftrightarrow \exists C \in \mathcal{C} / \{i, j\} \in C, \|i - j\|_2 = \sqrt{p} \end{aligned} \quad (14)$$

Thereby, it is possible to express the local probability law for each site $s \in S$:

$$p_s(l_s/x_s, l^s) = \frac{p_s(l_s/x_s, v_s) \exp(-U_s(x_s, l_s))}{Z_s(x_s, v_s)} \quad (15)$$

with $l^s = \{l_s/s \in S - \{s\}\}$, $Z(x_s, v_s) = \sum_{l_s \in \Omega_l} \exp(-U_s(x_s, l_s))$ and

$$U_s(x_s, l_s) = \Phi^1(x_s, l_s) + \Phi(l_s) + \Phi^P(l_s) \quad (16)$$

where

$$\begin{aligned} \Phi^P(l_s) &= \beta \sum_{r \in v_s: \langle s, r \rangle_1} (1 - \delta(l_s, l_r)) \\ &+ \frac{\beta}{\sqrt{2}} \sum_{r \in v_s: \langle s, r \rangle_2} (1 - \delta(l_s, l_r)) \end{aligned} \quad (17)$$

Φ must be chosen in order to respect the context of applicability of MCMC methods like SA [7]. Moreover, SA implies the introduction of a temperature T in the Gibbs distribution. The local probability can be written as follows:

$$p_s(l_s/x_s, l^s) = \frac{p_s(l_s/x_s, v_s) \exp(-U_s(x_s, l_s)/T)}{Z_s(x_s, v_s, T)} \quad (18)$$

Φ will allow the integration of geometrical and/or topological informations, *i.e.* "global" informations, about the segmented field at each iteration. Next part is an introduction to the principle of topological maps.

3 Topological map

Topological maps allow to represent the n -Dimension (n -D) region-segmented images. They encode both topology and geometry of images at the same time. Topological maps are combinatorial maps with particular properties. We begin this part by recalling the notion of combinatorial map before presenting the topological map's structure.

3.1 Combinatorial maps

Combinatorial maps were introduced in the seventies by [18], first as a planar graph representation model. They were extended by [19] in dimension n to represent orientable or non-orientable quasi-manifolds. They encode space subdivisions and all incidencey relationships. They are made of abstract elements, called *darts*, to which are defined applications, called α_i .

The combinatorial map definition in n -D is [20]:

Definition 1 (combinatorial maps) Let $n \geq 0$. A n combinatorial map, (or n -map) is an $(n + 1)$ -tuple $M = (B, \alpha_1, \dots, \alpha_n)$ where :

1. B is a finite set of darts;
2. α_1 is a permutation on B ;
3. $\forall 2 \leq i \leq n, \alpha_i$ is an involution on B ;
4. $\forall 1 \leq i \leq n - 2, \forall i + 2 \leq j \leq n, \alpha_i \circ \alpha_j$ is an involution.

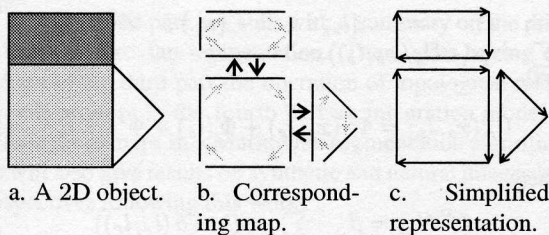


Figure 1: An object and the corresponding combinatorial map represented two different ways.

In this definition, there is an application α_i for each space dimension that puts in relationship two i dimension oriented cells. When two darts are linked by α_i , they are said to be α_i -sewed. Each space cell is implicitly represented by a set of darts. Figure 1.a shows an image whose corresponding combinatorial map is represented on figure 1.b. Each dart is represented by a segment, the α_1 relationship by light grey arrows and the α_2 relationship by dark grey arrows. α_1 puts in relationship a dart and the next dart of a same face.

For example, the light grey face of the image is represented by four α_1 -sewed darts on the map. The adjacency between this face and the dark grey face is represented by two darts α_2 -sewed together. We are using the simplified representation (figure 1.c) that does not explicitly represent the applications, to make it more understandable.

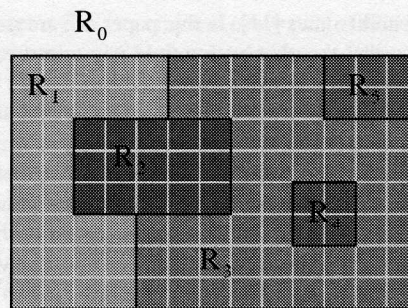
3.2 Topological map

A topological map's construction needs a region-based partition of the image, as opposed to its class-based partition we get from the markovian segmentation algorithm. So its construction will be preceded by a region growing algorithm, introduced to split all regions belonging to the same class.

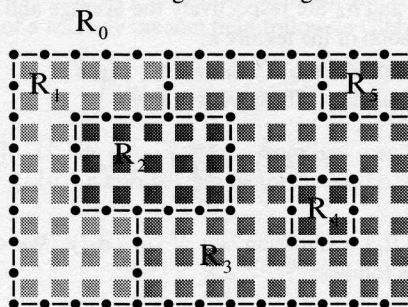
The topological map's construction consists in taking into account each pixel's border elements, to coherently define the topology of an image: the space is subdivided in elements and border elements, with which are provided incidence and adjacency relationships. Figure 2 represents a region-based segmented image and its interpixel borders. [14, 21] propose to redefine a simpler way the 2-D topological map introduced by Luc Brun in [22], in order to make it easily expendable to n -D. This definition uses simplification levels as follows:

1. Level 0: Full map

It is the starting point of the process. This map doesn't code the image's borders but simply all its interpixel elements. It is composed by squared faces, each of them represent a pixel of the image (figure 3.a).



a. A segmented image



b. Its regions' interpixel borders

Figure 2: Interpixel Vision

2. Level 1: Linel map

Level 1 map is obtained from the level 0 map and encodes the image's interpixel borders. This is done by removing every edge of that level 0 that doesn't belong to a border line of the image (figure 3.b).

3. Level 2: Borders map

It simplifies the linel map, by using edges to represent parts of a straight-line border instead of interpixels belonging to a border line of the image (figure 3.c).

4. Level 3: Topological map

From this last simplification level results the topological map, which now associates a single edge to each border (figure 3.d).

This last level is the topological map that encodes all topological and geometrical informations about the represented image. Moreover, it is minimal, stable for rigid transformations and unique. Such properties ensure that this model provides an efficient way to encode informations contained in an image. We are going to use this model to extract global informations, for instance the size or shape of regions, the number of adjacent regions, regions totally included into another...

The following part proposes an exploitation of some of these informations.

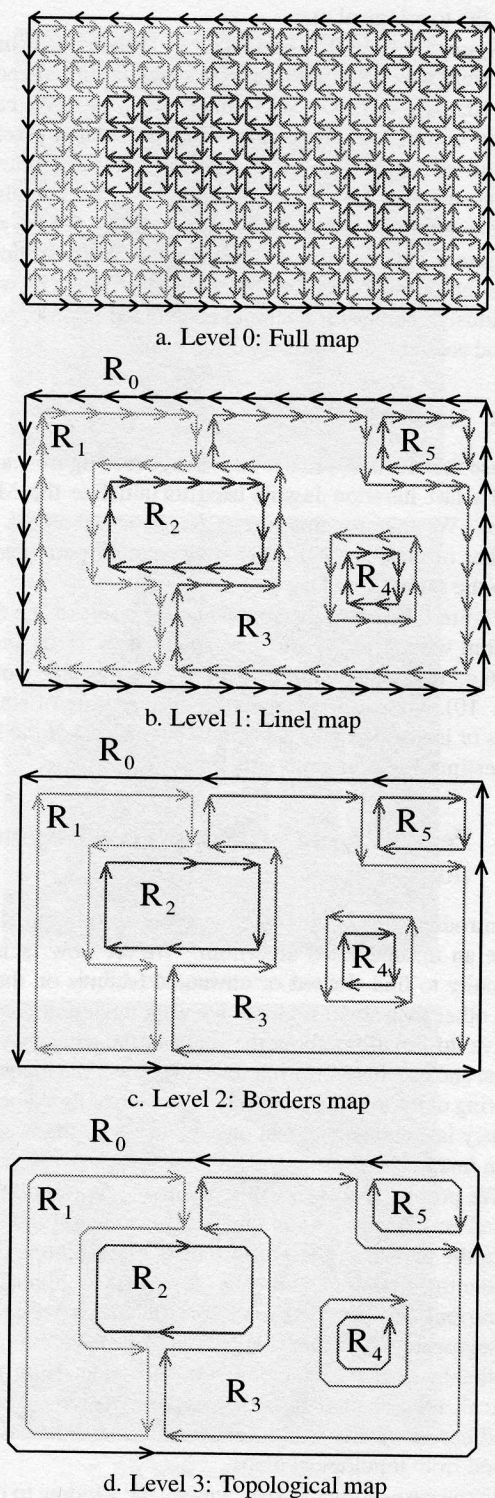


Figure 3: Topological map building

4 Integration of geometrical and topological informations

4.1 Size potential

The main purpose of this work is to make an algorithm able to segment any image in an unsupervised way. So we are looking for wanted or unwanted geometrical or topological features that could appear in the estimated label field.

One of these unwanted features is the presence of isolated pixels or small regions. Therefore, as a first potential for Φ (see (15) and (16)), we propose a “size” potential, Φ_{size} : it imposes a minimal size, q_{min} , below which the probability of pixel s to belong to a region will be reduced. q_{min} thresholding can be chosen to be proportional to the size of the image (see 4.2).

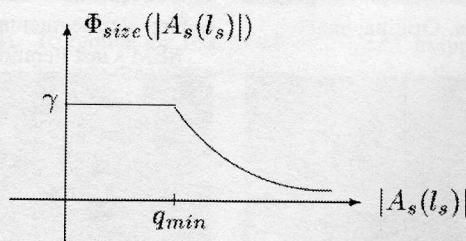


Figure 4: Φ_{size} size potential evolution in function of region size

A site's membership to a region is disfavored by minimizing its *a posteriori* probability to belong to that region (see (18)), or maximizing the U energy of the Gibbs probability (see (16)). Thereby, the chosen “size” potential function (figure 4) can be chosen as follows:

$$\Phi_{size}(|A_s(l_s)|) = \gamma (1 + \mathbf{1}_{(|A_s(l_s)| > q_{min})} \times (\exp(-\kappa(|A_s(l_s)| - q_{min})) - 1)) \quad (19)$$

$$\text{with } \mathbf{1}_{(|A_s(l_s)| > q_{min})} = \begin{cases} 1 & \text{if } |A_s(l_s)| > q_{min} \\ 0 & \text{otherwise} \end{cases} \quad \text{and}$$

$A_s(l_s)$ is the connected set of sites r such as $l_r = l_s$, i.e. the region to which the site s belongs. γ and κ are two constants that represent respectively the weight of the potential in the Gibbs energy function and the exponential decrease of Φ_{size} .

4.2 Segmentation results

4.2.1 Synthetic image

Figure 5.a shows a synthetic (32×32) image which is a realization of 6 Gaussian i.i.d. processes. Figure 5.b represents the last iteration of the SEM algorithm. The number

of classes and model parameters have been correctly estimated using $K_{max} = 10$ as the initial number of classes. In MCMC methods, the final result of the Markov Chain process does not depend on the initial law of X . So, we use the last iteration of the SEM algorithm as the initialization of the SA. We start from an initial temperature $T_0 = 1$ and use the classical decrease scheme: $T_k = 0.99^k T_0$. First we take $\beta = 0.5$ (Figure 5.c and 5.d).

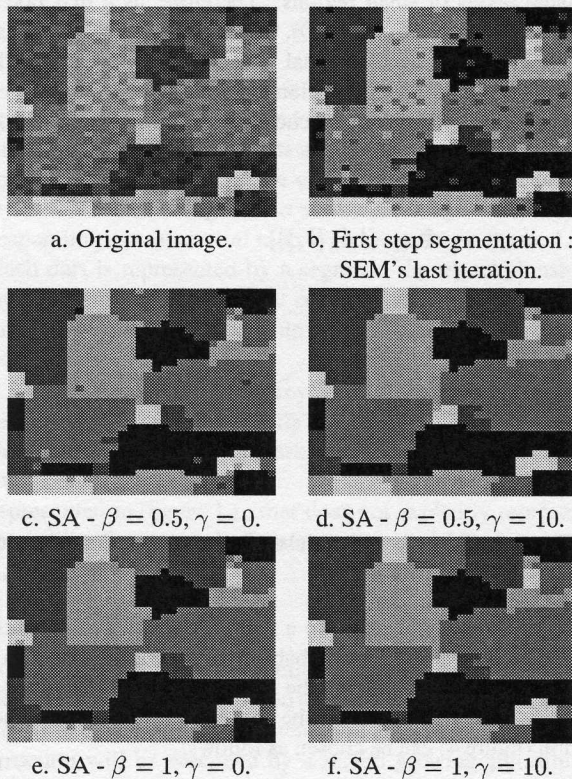


Figure 5: Synthetic image

Figure 5.c shows the result of the SA after 200 iterations without taking into account the potential on regions ($\gamma = 0$). We can note a few incoherences on this first segmented image: badly-defined borders, a few “isolated” pixels, that don’t seem meaningless when looking at the original image, but can be considered as a noise on the final partition.

Figure 5.e shows the result of the same experiment using this time $\beta = 1$. Increasing β is not a solution since it will produce a decrease in the accuracy of boundary estimation. High β values will provide a smoothing on region boundaries in the segmentation. However, additional informations about the image’s regions, such as their size, or adjacency, could be used to perform such kinds of enhancements. These informations, we call “global”, as opposed to

our previously used “local” informations, can be extracted from the topological map.

Figure 5.d ($\beta = 0.5$) and 5.f ($\beta = 1$) show the estimated label field using SA with the size potential initialized with $\gamma = 10$, $q_{min} = 0.002 \times |S|$ and $\kappa = \frac{3}{2 \times q_{min}}$. We can notice on both figures the disappearing of isolated pixels and yet still exact boundaries. So the size potential allows an enhancement of the segmentation process. We can also notice that although β has changed from figure 5.d ($\beta = 0.5$) to figure 5.f ($\beta = 0.5$ and $\beta = 1$), the final partition still looks the same. This is an interesting fact, since as we said previously, increasing β should be avoided in order to keep a good accuracy of boundary estimation.

4.2.2 Natural image

We use the “House” (128×128) image (see figure 6.a). The SEM’s last iteration is still used to initialize the Markov Chain. We take for this image $K_{max} = 20$ as the initial number of classes. All other segmentation parameters remain the same ($\beta = 1$).

Figure 6.b shows the segmentation result of the SA algorithm without using the size potential ($\gamma = 0$) and, figure 6.c, the segmentation result using the size potential ($\gamma = 10$). We can note again the disappearing of small regions or isolated pixels, especially on the wall of the house. The estimated boundaries still are correct.

4.3 Perspectives : semi-supervised segmentation

To introduce this part, we have evoked our intention to make an unsupervised algorithm. We are now facing the difficulty to find wanted or unwanted features on the label field other than small regions: we want the topological map to give informations about the shape or the adjacency of regions, and use these informations to segment the image. But, favoring or unfavoring a region in function of its shape or adjacency is a choice that can only be made by the user, for a given image.

We propose two examples: an image representing geological cuts (Figure 7) and the “House” image (Figure 6.a).

In the first case, the purpose of segmentation is to keep the identified geological breaks (see 7.b). Our algorithm, in the current unsupervised version, is unable to get a satisfying segmentation of that image. But if we decide to use the fact that breaks represent relatively long and narrow, mainly vertical, regions, and favor this type of regions, the result could become convincing. Such attributes can be easily extracted from topological maps.

In the case of the “House” image, the window to our left is a rather small region that could disappear during the segmentation, although we would like to keep it. This region has an interesting propriety: it is adjacent to only one region



a. Original image

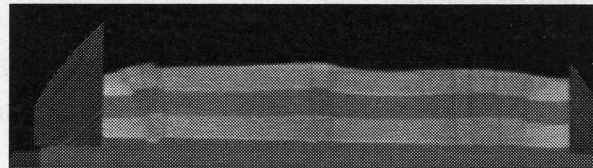


b. SA without size potential.

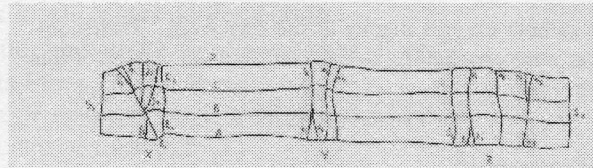


c. SA with size potential.

Figure 6: Segmentations of “House” image



a. Geological cut



b. Geological breaks

Figure 7: Geological cut

(the wall). To reduce the risk to see this region disappear, we have to provide the information to favor regions that are adjacent to only one region. We could also use the fact that most regions have rectilinear borders to get a better segmentation. Once again here are two informations a topological map can give us without any difficulty.

In this paper, we limited our presentation to the unsupervised case. We are currently working on the semi-supervised segmentation method as discussed above.

5 Conclusion

The proposed algorithm shows up as a Markovian relaxation algorithm to which are added global informations. These informations are extracted from the topological map and its intermediate building levels. They are added as energy potentials thanks to the Gibbs distribution. Moreover, our segmentation algorithm is unsupervised considering the number of classes and their model parameters. Continuing in an unsupervised way, we introduced the “size potential” to avoid isolated pixels or small regions. Our approach provided accurate results on a synthetic image and a natural image.

Working in an unsupervised way limits the use of global informations. The next step will consist in providing an user interface, on which choices are proposed, such as: “favor long and narrow regions”, “favor vertical regions”, “keep regions adjacent to one single region”, etc. This we defined as a semi-supervised approach.

Furthermore, in particular cases, we could also consider removing the learning phase SEM. Model parameters would then be estimated by a specific learning phase like Viterbi algorithm. The segmentation would only consist in a relaxation algorithm. This would turn into a supervised segmentation method.

There are many topological or geometrical informations that can be useful to a segmentation process. We are currently working on the definition of criteria based on some of these informations. These criteria should improve results on image segmentation.

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